CHAPTER 25

CANONICAL EQUATION K_{25} FOR RANDOM G-MATRICES. STRONG V-LAW

We present a survey of some recent results established for non-Hermitian random matrices and propose a new theory of these matrices based on the V-transform of the normalized spectral functions (n.s.f.) $\nu_n(x, y)$ of the eigenvalues of a nonsymmetric matrix Ξ via the n.s.f. $\mu_n(x, t, s)$ of the eigenvalues of the Hermitian matrix $(\Xi_n - \tau I)(\Xi_n - \tau I)^*$, $\tau = t + is$. We determine the general form of possible limit normalized spectral functions of the matrix Ξ_n, i.e., prove the so-called V-law. We have now a great deal of evidence that the V-law has many applications, especially in physics.

25.1. Formulation of the main assertion

We first formulate our main result.

Theorem 25.1. (See the V-equation in [Gir72], [Gir73], [Gir84], [Gir92], [Gir96]). *Let $\Xi_n = (\xi_{ij})_{i,j=1}^{n}$ be a complex random matrix whose entries $\xi_{ij}^{(n)}, i, j = 1, ..., n$, are independent for every n and given in the common probability space,*

$$\mathbf{E}\, \xi_{ij}^{(n)} = a_{ij}^{(n)},\ \mathbf{E} \left| \xi_{ij}^{(n)} - a_{ij}^{(n)} \right|^2 = \sigma_{ij}^{(n)}, i, j = 1, ..., n, \tag{25.1}$$

$$\sup_n \max_{\substack{i=1,\ \dots\ ,\ n,\\ j=1,\ \dots\ ,\ n}} \left\{ \sum_{j=1}^{n} \sigma_{ij}^{(n)} + \sum_{i=1}^{n} \sigma_{ij}^{(n)} + \sum_{j=1}^{n} \left| a_{ij}^{(n)} \right| + \sum_{i=1}^{n} \left| a_{ij}^{(n)} \right| \right\} < \infty, \tag{25.2}$$

$$\sigma_{pl}^{(n)} n > c > 0\ p, l = 1, ..., n,\ n = 1, 2, ..., \tag{25.3}$$

and, for some $\delta > 0$,

$$\max_{p,l=1,\dots,n} \mathbf{E} \left| \left(\xi_{pl}^{(n)} - a_{pl}^{(n)} \right) \sqrt{n} \right|^{2+\delta} \le c < \infty. \tag{25.4}$$

Suppose that the densities $p_{kl}^{(n)}(x)$ of real or imaginary parts of the entries $\sqrt{n}\xi_{pl}^{(n)}$ exist and are such that

$$\sup_n \max_{k=1,\dots,n} \int \left[p_{kk}^{(n)}(x) \right]^{\beta} \mathrm{d}x < \infty \quad \text{or} \quad \sup_n \max_{k=1,\dots,n} \int \left[q_{kk}^{(n)}(x) \right]^{\beta} \mathrm{d}x < \infty, \tag{25.5}$$

where $\beta > 1$ is a certain number.

Then, with probability one,

$$\lim_{\alpha \downarrow 0} \lim_{n \to \infty} \left| \mu_n \left(x, y, \Xi_n \right) - F_{n,\alpha} \left(x, y \right) \right| = 0, \tag{25.6}$$

where

$$\mu_n \left(x, y, \Xi_n \right) = n^{-1} \sum_{k=1}^{n} \chi \left\{ \operatorname{Re} \lambda_k < x, \operatorname{Im} \lambda_k < y \right\},$$

λ_k are eigenvalues of the matrix Ξ_n, the V-density

$$p_{n,\alpha}(t,s) = \frac{\partial^2}{\partial x \, \partial y} F_{n,\alpha}(x,y)$$

is equal to

$$p_{n,\alpha} \left(t, s \right) = \begin{cases} -\frac{1}{4\pi} \int_{\alpha}^{\infty} \left[\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2} \right] m_n \left(y, t, s \right) \mathrm{d}y & \text{for} \ \ (t,s) \notin G, \\ 0 & \text{for} \ \ (t,s) \in G, \end{cases} \tag{25.7}$$

$$m_n \left(y, t, s \right) = \frac{1}{n} \operatorname{Tr} \ \left[C_1 + \left(A_n - \tau I_n \right) C_2^{-1} \left(A_n - \tau I_n \right)^* \right]^{-1}, \ y > 0, \tag{25.8}$$

$$A_n = \left(a_{ij} \right)_{i,j=1}^{n}, \quad C_1 = \left(c_{1i}(y) \delta_{ij} \right)_{i,j=1}^{n}, \quad C_2 = \left(c_{2i}(y) \delta_{ij} \right)_{i,j=1}^{n},$$

$c_{2i}(y)$ and $c_{1i}(y), i = 1, ..., n$ satisfy the system of V-equations K_{25} for G-matrices:

$$\begin{cases} c_{1p}(y) = y + \sum_{j=1}^{n} \sigma_{pj}^{(n)} \left\{ \left[C_2 + \left(A_n - \tau I_n \right)^* C_1^{-1} \left(A_n - \tau I_n \right) \right]^{-1} \right\}_{jj}, \\ c_{2k}(y) = 1 + \sum_{j=1}^{n} \sigma_{jk}^{(n)} \left\{ \left[C_1 + \left(A_n - \tau I_n \right) C_2^{-1} \left(A_n - \tau I_n \right)^* \right]^{-1} \right\}_{jj}, \end{cases} \tag{25.9}$$

$p, k = 1, ..., n$, there exists a unique solution of this system of equations in the class of real positive analytic functions in $y > 0$, and the V-region G is equal to

$$G = \left\{ (t,s) : \ \limsup_{\alpha \downarrow 0} \limsup_{n \to \infty} \left| \left(\partial / \partial \alpha \right) m_n \left(\alpha, t, s \right) \right| < \infty \right\}. \tag{25.10}$$