CHAPTER 21

CANONICAL EQUATION K_{21} FOR RANDOM MATRICES WITH INDEPENDENT PAIRS OF ENTRIES WITH ZERO EXPECTATIONS. CIRCULAR AND ELLIPTIC LAWS

In this chapter, we establish the canonical equation K_{21}, which is in remarkable relationship with the so-called elliptic and circular laws discovered in 1982 in [Gir27].

21.1. Basic Equation

We consider complex matrices $Q = \Xi_n - \tau I_n$, where $\Xi_n = \left\{\xi_{ij}^{(n)}\right\}_{i,j=1}^{n}$, $\tau = t + \mathrm{i}s$, and the G-function

$$\mu_n(x,t,s) = n^{-1}\sum_{k=1}^{n}\chi\{\lambda_k(QQ^*) < x\},$$

where $\lambda_i(QQ^*)$ are eigenvalues of the random matrix QQ^*.

Theorem 21.1 [Gir 84]. *Let $\Xi = (\xi_{ij}^{(n)})_{i,j=1,\dots,n}$ be complex random matrices such that the pairs of their entries $\left(\xi_{ij}^{(n)},\ \xi_{ji}^{(n)}\right)$, $i \geq j$, $i,j = 1,\dots,n$, are independent for all n, $\mathbf{E}\,\xi_{ij}^{(n)} = 0$, $\mathbf{E}\,|\xi_{ij}^{(n)}|^2 = n^{-1}$,*

$$\mathbf{E}\,\xi_{ij}^{(n)}\xi_{ji}^{(n)} = \rho n^{-1},\ i \neq j,\ i,j = 1,\dots,n,$$

where ρ is a complex number and $\xi_{ij}^{(n)}$ are defined on a common probability space, and, for some $\delta > 0$,

$$\sup_n \max_{i,j=1,\dots,n} \mathbf{E}\left|\left[\xi_{ij}^{(n)} - a_{ij}^{(n)}\right]\sqrt{n}\right|^{2+\delta} \leq c < \infty.$$

Then, with probability one, for any τ and almost all x,

$$\lim_{n\to\infty}|\mu_n(x,t,s)) - F(x,t,s)| = 0,$$

where $F(x,t,s)$ is a distribution function in x whose Stieltjes transform $b(\alpha) = \int_0^\infty (\alpha + x)^{-1}\,\mathrm{d}_x F(x,t,s)$, $\alpha > 0$, is equal to the solution $b(\alpha)$ of the canonical equation K_{21}

$$b(\alpha) = \frac{1}{2\pi}\int_{-2}^{-2}\sqrt{4-u^2}\left[\alpha\left(1+(1-|\rho|)\,b\left(\alpha\right)\right)+\frac{\left|\tau-u\sqrt{\rho}\right|^2}{1+(1-|\rho|)\,b\left(\alpha\right)}\right]^{-1}\mathrm{d}u.$$

The equation K_{21} is uniquely solvable in the class of real nonnegative analytic functions $b\left(\alpha\right), \alpha > 0$.

21.2. Elliptic Law

The convenience of the canonical equation K_{21} becomes clear from the proof of the following brilliant result in the theory of random matrices:

Theorem 21.2 (Elliptic Law) [Gir 84]. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a random complex matrix $\Xi = (\xi_{ij}^{(n)})_{i,j=1}^{n}$ such that the pairs of entries $\{\xi_{ij}^{(n)}, \xi_{ji}^{(n)}\},\ i \geq j,\ i,j = 1,\ldots,n$, are independent for each n,*

$$\mathbf{E}\,\xi_{ij}^{(n)} = 0,\quad \mathbf{E}\,|\xi_{ij}^{(n)}|^2 = n^{-1},\quad \mathbf{E}\,\xi_{ij}^{(n)}\xi_{ji}^{(n)} = \rho n^{-1},\ i \neq j,$$

where ρ is a complex number, and, for some $\delta > 0$,

$$\sup_n \max_{i,j=1,\ldots,n} \mathbf{E}\,|\xi_{ij}^{(n)}\sqrt{n}|^{4+\delta} \leq c < \infty.$$

Let $a = \mathrm{Re}\sqrt{\rho},\ b = \mathrm{Im}\sqrt{\rho}$ and $G = \{(u,v):\ u^2+v^2 < 1\}$ for $|\rho| = 0$, and

$$G = \left\{(u,v):\ \frac{(bu-a\nu)^2}{(1-|\rho|)^2} + \frac{(au+b\nu)^2}{(1+|\rho|)^2} < |\rho|\right\} \text{ for } 0 < |\rho| < 1.$$

Also let

$$\nu_n(x,y) = n^{-1}\sum_{k=1}^{n}\chi(\mathrm{Re}\,\lambda_k < x)\chi(\mathrm{Im}\,\lambda_k < y,)$$

where χ is the indicator function, be a normalized spectral function. Then

$$\underset{n\to\infty}{p\lim}\begin{cases} \sup\limits_{x,y}\left|\nu_n(x,y) - \frac{1}{\pi(1-|\rho|^2)}\displaystyle\int_{\{u<x;\,\nu<y\}\cap G}\mathrm{d}u\,\mathrm{d}\nu\right| = 0, & \text{for } 0 \leq |\rho| < 1, \\ \sup\limits_{x,y}\left|\nu_n(x,y) - \frac{\chi(y<0)}{2\pi}\displaystyle\int_{\{u<x\}\cap\{|u|<2\}}\sqrt{4-u^2}\,\mathrm{d}u\right| = 0, & \text{for } b = 0,\ |\rho| = 1, \\ \sup\limits_{x,y}\left|\nu_n(x,y) - \frac{\chi(x<0)}{2\pi}\displaystyle\int_{\{\nu<y\}\cap\{|\nu|<2\}}\sqrt{4-\nu^2}\,\mathrm{d}\nu\right| = 0, & \text{for } a = 0,\ |\rho| = 1. \end{cases}$$