## CHAPTER 20

## CANONICAL EQUATION $K_{20}$. STRONG LAW FOR NORMALIZED SPECTRAL FUNCTIONS OF NONSELFADJOINT RANDOM MATRICES WITH INDEPENDENT ROW VECTORS. SIMPLE RIGOROUS PROOF OF THE STRONG CIRCULAR LAW

Mathematical models which are described by non–Hermitian random matrices have recently attracted a lot of attention. In the present chapter, we proceed to the analysis of normalized spectral functions of nonsymmetrical random matrices. For the complex random matrix such that the real and imaginary parts of its entries are independent and distributed according to standard normal law the Circular distribution for the expected normalized spectral functions $\mathbf{E}\,\nu_n\left(u,v,\sqrt{\sigma^2 2^{-1}}\Xi n^{-1/2}\right)$ was proved in [Gin] and [Meh1]. As reader knows a law is a mathematical statement which always holds true. Do not confuse Circular distribution with Circular law which take place for the wide class of random matrices. Some explanations and generalized formulas for the proof of the strong Circular law published in [Gir33] are presented: Let $\Xi = \left(\xi_{ij}^{(n)}\right)_{i,j=1}^{n}$ be a complex random matrix whose entries $\xi_{ij}^{(n)}$ defined on a common probability space are independent for any $n = 1, 2, ...$ and such that $\mathbf{E}\,\xi_{ij}^{(n)} = 0$ and $\mathbf{E}\left|\xi_{ij}^{(n)}\right|^2 = n^{-1}$. Further, assume that there exist densities $p_{ij}^{(n)}(x)$ of the real parts (or densities $q_{ij}^{(n)}(x)$ of the imaginary parts) of the random entries $\sqrt{n}\xi_{ij}^{(n)}$ and, for some $\delta > 0$ and $\beta > 1$

$$\sup_n \max_{i,j=1,\dots,n} \mathbf{E}\left|\xi_{ij}^{(n)}\sqrt{n}\right|^{2+\delta} \leq c < \infty,$$

$$\sup_n \max_{k,l=1,\dots,n} \int_{-\infty}^{\infty} \left[p_{kl}^{(n)}(x)\right]^{\beta} \mathrm{d}x \leq c < \infty,$$

(or $\sup\limits_n \max\limits_{k,l=1,\dots,n} \int_{-\infty}^{\infty} \left[q_{kl}^{(n)}(x)\right]^{\beta} \mathrm{d}x \leq c < \infty.$ )

Then, with probability one,

$$\lim_{n\to\infty} \mu_n(x,y) = \pi^{-1} \iint\limits_{\{u<x,\ v<y\}\cap\{u^2+v^2<1\}} \mathrm{d}u\,\mathrm{d}v,$$

where $\mu_n(x,y) = n^{-1}\sum\limits_{k=1}^{n} \chi(\operatorname{Re}\lambda_k < x)\chi(\operatorname{Im}\lambda_k < y)$ is a normalized spectral function and $\lambda_k$ are eigenvalues of a complex random matrix $\Xi = \left(\xi_{ij}^{(n)}\right)_{i,j=1}^{n}$.

### 20.8. Rigorous proof of the strong circular law

On the basis of the results established in the previous section and Theorem 20.6, we now formulate our general result.

**Theorem 20.7** (see [Gir33], [Gir54, p.428], [Gir87]). *For any $n$, let the random entries $\xi_{pl}^{(n)}$, $l, p = 1, \dots, n$, of a complex matrix $H_n = (\xi_{pl}^{(n)} n^{-1/2})$ be independent and defined on a common probability space, $\mathbf{E}\, \xi_{pl}^{(n)} = 0$, $\mathbf{E}\, |\xi_{pl}^{(n)}|^2 = \sigma^2$, $0 < \sigma < \infty$. Assume that either the densities of the real parts $p_{ij}^{(n)}(x)$ or the densities of the imaginary parts $q_{ij}^{(n)}(x)$ of random entries $\xi_{ij}^{(n)}$ exist and satisfy the condition,*

$$\sup_n \max_{k,l=1,\dots,n} \int_{-\infty}^{\infty} \left[p_{kl}^{(n)}(x)\right]^{\beta} \mathrm{d}x \le c < \infty, \ \beta > 1,$$

*or*

$$\sup_n \max_{k,l=1,\dots,n} \int_{-\infty}^{\infty} \left[q_{kl}^{(n)}(x)\right]^{\beta} \mathrm{d}x \le c < \infty, \ \beta > 1,$$

*for some $\delta > 0$, and*

$$\sup_n \sup_{p,l=1,\dots,n} \mathbf{E}\, |\xi_{pl}^{(n)}|^{2+\delta} < \infty.$$

*Then, for any $x$ and $y$, with probability 1*

$$\lim_{n\to\infty} \nu_n(x, y, H_n) = \nu(x, y),$$

*where*

$$\frac{\partial^2 \nu(x,y)}{\partial x\, \partial y} = \begin{cases} \sigma^{-2}\pi^{-1} & \text{for } x^2 + y^2 < \sigma^2, \\ 0 & \text{for } x^2 + y^2 \ge \sigma^2, \end{cases}$$

$$\nu_n(x, y, H_n) = n^{-1} \sum_{k=1}^{n} \chi(\mathrm{Re}\, \lambda_k < x)\chi(\mathrm{Im}\, \lambda_k < y),$$

*$\lambda_k$ are eigenvalues of the matrix $H_n$.*