Estimation of states of some systems which describes by recursive equations. Spectral Equation S_7

V. L. GIRKO

Department of Probability and Statistics, Michigan State University
East Lansing, Michigan 48824

Let us consider the problem of estimating of states of systems with discrete time which describes by the equation of state of some system

$$\vec{x}_{k+1} = \vec{x}_k + A_k\vec{x}_k + \vec{\xi}_k; \quad \vec{y}_k = C_k\vec{x}_k + \vec{\eta}_k; \quad k = 0, 1, 2, ... \tag{12.1}$$

where $\vec{x}_k$ are the n-dimensional vectors of state; $\vec{y}_k$ p-dimensional vectors of observed of exit variables; A_k are square $n \times n$ matrix; C_k are matrices of dimension $p \times n$; $\vec{\xi}_k$ and $\vec{\eta}_k$ are some vectors of obstacles which have dimensions n and p respectively and which satisfying the condition

$$\vec{\xi}_i \in G, \quad G = \{\vec{\xi}_i : \sum_{i=0}^{k} \|\vec{\xi}_i\|^2 \leq 1\}.$$

$$\mathbf{E}\,\vec{\eta}_i = 0, \quad \mathbf{E}\,\vec{\eta}_i\vec{\eta}_i^T = R_i.$$

In this article we consider the following problem of estimating of state $\vec{x}_k$: to find matrices K_i of dimension $n \times p$ and vector $\vec{l}$ of dimension n which minimized the expression

$$\max_{\vec{\xi}_i \in G} \mathbf{E}\left(\vec{x}_{k+1} - \sum_{i=0}^{k} K_i\vec{y}_i - \vec{l}\right)^T V\left(\vec{x}_{k+1} - \sum_{i=0}^{k} K_i\vec{y}_i - \vec{l}\right), \tag{12.2}$$

where V is a nonnegative definite matrix of sizes $n \times n$.
Let L_1 is the set of matrices of dimension $n \times p$; L_2 is the set of vectors $\vec{l}$ of dimension n.

Theorem 12.1. *Under above formulated assumptions*

$$\begin{aligned} &\max_{\vec{\xi}_i \in G} \mathbf{E}\left(\vec{x}_{k+1} - \sum_{i=0}^{k} K_i\vec{y}_i - \vec{l}\right)^T V\left(\vec{x}_{k+1} - \sum_{i=0}^{k} K_i\vec{y}_i - \vec{l}\right) \\ &= \lambda_1\left\{\sum_{i=1}^{k} Z_{i+1}VZ_{i+1}^T\right\} + \sum_{i=1}^{k} \mathrm{Tr} R_i K_i^* V K_i^{*T}, \end{aligned} \tag{12.3}$$

where the matrices Z_i satisfy the recursion equation

$$Z_{p+1}(I + Z_p) = Z_p - K_p^* C_p; \; p = 0, \dots, k; \; Z_{k+1} = I; \; \vec{l}^* = Z_0 x_0;$$

the matrices K_i^* satisfy the equations

$$-\sum_{i=0}^{s} p_i C_j S_j \vec{\varphi}_i \vec{\varphi}_i^T + R_j K_j^* V = 0; \quad j = 1, \ldots, k; \tag{12.4}$$

where $p_i > 0;\ \ \sum_{i=0}^{s} p_i = 1$ *are arbitrary variables,* $\vec{\varphi}_k,\ \ k = 1, \ldots, s$ *are orthonormal eigenvectors which correspond to the maximal s-multiple eigenvalue* λ_1 *of the matrix* $\sum_{i=1}^{k} Z_{i+1} V Z_{i+1}^T$ *, the matrices* S_p *satisfy the system of equations*

$$S_{p+1} = S_p + A_p S_p - V Z_{p+1}^T;\ p = 1, \ldots, k;\ S_1 = 0. \tag{12.5}$$

Proof. It is obvious that

$$\begin{aligned}
&\big(\vec{x}_{k+1} - \sum_{i=0}^{k} K_i \vec{y}_i - \vec{l}\big)^T V \big(\vec{x}_{k+1} - \sum_{i=0}^{k} K_i \vec{y}_i - \vec{l}\big) \\
&= \big(\vec{x}_{k+1} - \sum_{i=0}^{k} K_i C_i \vec{x}_i - \sum_{i=0}^{k} K_i \vec{\eta}_i - \vec{l}\big)^T V \big(\vec{x}_{k+1} - \sum_{i=0}^{k} K_i C_i \vec{x}_i - \sum_{i=0}^{k} K_i \vec{\eta}_i - \vec{l}\big)^T
\end{aligned}$$

Let us consider the system of recursion equations

$$Z_{p+1} = Z_p - Z_{p+1} A_p + K_p C_p; \quad p = 0, ;k$$

with initial condition $Z_{k+1} = I$. Then using (12.1) after obvious transformations we have

$$\begin{aligned}
\vec{x}_{k+1} - \sum_{i=0}^{k} K_i C_i \vec{x}_i &= \vec{x}_{k+1} - \sum_{i=0}^{k} (Z_{i+1} - Z_i) \vec{x}_i - \sum_{i=0}^{k} Z_{i+1} A_i \vec{x}_i \\
&\quad + \vec{x}_{k+1} - \sum_{i=0}^{k} Z_{i+1} (\vec{x}_{i+1} - A_i \vec{x}_i - \vec{\xi}) \\
&\quad - \sum_{i=0}^{k} Z_{i+1} A_i \vec{x}_i + \sum_{i=0}^{k} Z_i \vec{x}_i \\
&= Z_0 \vec{x}_0 + \sum_{i=0}^{k} Z_{i+1} \vec{\xi}_i.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(\vec{x}_{k+1} - \sum_{i=0}^{k} K_i \vec{y}_i - \vec{l})^T V (\vec{x}_{k+1} - \sum_{i=0}^{k} K_i \vec{y}_i - \vec{l})^T \\
&= (Z_0 \vec{x}_0 + \sum_{i=0}^{k} Z_{i+1} \vec{\xi}_i + \sum_{i=0}^{k} K_i \vec{\eta}_i - \vec{l})^T V (Z_0 \vec{x}_0 + \sum_{i=0}^{k} Z_{i+1} \vec{\xi}_i + \sum_{i=0}^{i} K_i \vec{\eta}_i - \vec{l})^T.
\end{aligned}$$

Using the subdifferential calculus [1] and (12.2) we get

$$\min_{K_i\in L_1;\vec{l}\in L_2}\max_{\vec{\xi}_i\in G}\mathbf{E}\left(\vec{x}_{k+1}-\sum_{i=0}^{k}K_i\vec{y}_i-\vec{l}\right)^T V\left(\vec{x}_{k+1}-\sum_{i=0}^{k}K_i\vec{y}_i-\vec{l}\right)$$
$$=\min_{K_i\in L_1}\left\{\lambda_1\left(\sum_{i=1}^{k}Z_{i+1}VZ_{i+1}^T\right)+\sum_{i=1}^{k}\mathrm{Tr}R_iK_i^*VK_i^{*T}\right\},\ \vec{l}^*=Z_0x_0$$

and unknown matrices K_i^* satisfy equation

$$\mathrm{Tr}\left\{\sum_{q=1}^{s}\vec{\varphi}_q\vec{\varphi}_q^T p_q\sum_{i=1}^{k}\tilde{Z}_{i+1}VZ_{i+1}^T+\sum_{i=1}^{k}\Theta_iR_iK_i^*V\right\}=0 \tag{12.6}$$

where Θ_i are arbitrary matrices which have the same dimension as matrices K_i and matrices $\tilde{Z}_{i+1}$ satisfy to the equations

$$\tilde{Z}_{i+1}=\tilde{Z}_i-\tilde{Z}_{i+1}A_i+\Theta_iC_i;\ \tilde{Z}_{k+1}=0;\ i=1,\dots,k.$$

Obviously

$$\sum_{i+1}^{k}\tilde{Z}_{i+1}VZ_{i+1}^T=\sum_{i=1}^{k}\tilde{Z}_{i+1}ZZ_{i+1}^T+(\tilde{Z}_{i+1}S_{i+1}-\tilde{Z}_iS_i)$$
$$=\sum_{i=1}^{k}(\tilde{Z}_{i+1}VZ_{i+1}^T+\tilde{Z}_{i+1}(S_i+A_iS_i-VZ_{i+1}^T)-\tilde{Z}_iS_i)$$

$$\sum_{i=1}^{k}[(\tilde{Z}_i-Z_{i+1}A_i+\Theta_iC_i)S_i+\tilde{Z}i+1A_iS_i-Z_{i+1}^T)]=\sum_{i=1}^{k}\Theta_iC_iS_i.$$

Using this equality and the auxiliary systems of equations (12.5) we obtain that (12.6) equals

$$\mathrm{Tr}\left\{\sum_{q=1}^{s}\vec{\varphi}_q\vec{\varphi}_q^T p_q\sum_{i=1}^{k}(\tilde{Z}_{i+1}VZ_{i+1}^T+\tilde{Z}_{i+1}S_{i+1}-\tilde{Z}y_iS_i)+\sum_{i=1}^{k}\Theta_iR_iK_i^*V\right\}$$
$$=\mathrm{Tr}\left\{-\sum_{q=1}^{s}\vec{\varphi}_q\vec{\varphi}_q^T p_q\sum_{i=1}^{k}\Theta_iC_iS_i+\sum_{i=1}^{k}\Theta_iR_iK_i^*V\right\}.$$

From this equation we obtain the all assertions of the theorem.

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