Estimation of states of some recursively defined systems. Estimator S_6

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Consider the problem of estimation of states of systems with discrete time described by the equations

$$\vec{x}_{i+1} = \vec{x}_i + A_i\vec{x}_i + \vec{\xi}_i\,; \quad \vec{y}_i = C_i\vec{x}_i + \vec{\eta}_i; \;\; i = 1, 2, ... \tag{11.1}$$

where $\vec{x}_i$ are the n-dimensional vectors of state; $\vec{y}_i$ are p-dimensional vectors of observed variables; A_i are square $n \times n$ matrices; C_i are matrices of dimension $p \times n$; $\vec{\xi}_i$ and $\vec{\eta}_i$ are error vectors of dimensions n and p, respectively, which satisfy the inequality

$$\vec{\xi}_i\,, \vec{\eta}_i \in G, \;\; G = \left\{ \vec{\xi}_i\,, \vec{\eta}_i : \sum_{i=1}^{k} \left(\left\| \vec{\xi}_i \right\|^2 + \|\vec{\eta}_i\|^2 \right) \le 1 \right\}.$$

In this section we generalize the estimator S_3 for such an equation. Consider the following problem of estimating the state $\vec{x}_k$: find matrices $\hat{K}_i$ of dimension $n \times p$ and a vector $\vec{l}$ of dimension n which minimize the expression

$$\max_{\vec{\xi}_i\,, \vec{\eta}_i \in G} \left\| \vec{x}_{k+1} - \sum_{i=1}^{k} K_i\vec{y}_i - \vec{l} \right\|^2 . \tag{11.2}$$

Let $L_{n\times p}$ be the set of real matrices of dimension $n \times p$ and let L_n be the set of real vectors $\vec{l}$ of dimension n.

THEOREM 11.1. *Under the above formulated assumptions*

$$\min_{K_i \in L_{n\times p}; \vec{l} \in L_n} \max_{\vec{\xi}_i\,, \vec{\eta}_i \in G} \left\| \vec{x}_{k+1} - \sum_{i=1}^{k} K_i\vec{y}_i - \vec{l} \right\|^2 = \lambda_1 \left\{ \sum_{i=1}^{k} \left(Z_{i+1}Z_{i+1}^T + K_i^* K_i^{*T} \right) \right\}, \tag{11.3}$$

where the matrices Z_i satisfy the recursive equations

$$Z_{i+1}\left(I + A_i\right) = Z_i + K_i^* C_i; \;\; p = 1, ..., k; \;\; Z_{k+1} = I; \;\; \vec{l}^{\,*} = Z_1\vec{x}_1;$$

the matrices $K_i^*; i = 1, ..., k$ satisfy the system of equations S_6

$$\sum_{l=1}^{s} p_l [K_p^{*T} + C_p S_p] \vec{\varphi}_l \vec{\varphi}_l^{\,T} = 0; \;\; p = 1, ..., k \tag{11.4}$$

where

$$p_l > 0,\ \ l = 1, ..., s;\ \ \sum_{j=0}^{s} p_l = 1,$$

$\vec{\varphi}_k,\ \ k = 1, ..., s$ *are orthonormal eigenvectors which correspond to the maximal* s*-multiple eigenvalue* λ_1 *of the matrix*

$$\sum_{i=1}^{k} \left[Z_{i+1} Z_{i+1}^T + K_i^* K_i^{*T} \right],$$

the matrices S_p *satisfying the system of equations*

$$S_{p+1} = S_p + A_p S_p - Z_{p+1}^T;\ \ p = 1, \cdots, k;\ \ S_1 = 0, \tag{11.5}$$

one of the solutions of equation (10.4) is

$$K_p^{*T} = -C_p S_p,\ \ p = 1, ..., k.$$

Proof. It is obvious that

$$\left\| \vec{x}_{k+1} - \sum_{i=1}^{k} K_i \vec{y}_i - \vec{l} \right\|^2 = \left\| \vec{x}_{k+1} - \sum_{i=1}^{k} K_i C_i \vec{x}_i - \sum_{i=1}^{k} K_i \vec{\eta}_i - \vec{l} \right\|^2.$$

Consider the system of recursive equations

$$Z_{p+1} = Z_p - Z_{p+1} A_p + K_p C_p;\ \ p = 1, ..., k$$

with the initial condition $Z_{k+1} = I$. Then, using (10.1) after obvious transformations we have

$$\begin{aligned} \vec{x}_{k+1} - \sum_{i=1}^{k} K_i C_i \vec{x}_i &= \vec{x}_{k+1} - \sum_{i=1}^{k} (Z_{i+1} - Z_i) \vec{x}_i - \sum_{i=1}^{k} Z_{i+1} A_i \vec{x}_i \\ &= \vec{x}_{k+1} - \sum_{i=1}^{k} Z_{i+1} (\vec{x}_{i+1} - A_i \vec{x}_i - \vec{\xi}_i) - \sum_{i=1}^{k} Z_{i+1} A_i \vec{x}_i + \sum_{i=1}^{k} Z_i \vec{x}_i \\ &= Z_1 \vec{x}_1 + \sum_{i=1}^{k} Z_{i+1} \vec{\xi}_i. \end{aligned}$$

Therefore

$$\left\| \vec{x}_{k+1} - \sum_{i=1}^{k} K_i \vec{y}_i - \vec{l} \right\|^2 = \left\| Z_1 \vec{x}_1 + \sum_{i=1}^{k} Z_{i+1} \vec{\xi}_i + \sum_{i=1}^{k} K_i \vec{\eta}_i - \vec{l} \right\|^2.$$

Using the proof of Theorem 5.1 we get

$$\min_{\substack{K_i \in L_{n\times p};\\ \vec{l}\in L_n}} \max_{\vec{\xi}_i,\vec{\eta}_i \in G} \left\| \vec{x}_{k+1} - \sum_{i=1}^{k} K_i \vec{y}_i - \vec{l} \right\|^2 = \min_{K_i \in L_{n\times p}} \lambda_1 \left\{ \sum_{i=1}^{k} \left(Z_{i+1} Z_{i+1}^T + K_i^* K_i^{*T} \right) \right\},$$
$$\vec{l}^* = Z_1 \vec{x}_1,$$

and the unknown matrices K_i^* satisfy the equation

$$\sum_{q=1}^{s} \vec{\varphi}_q^T p_q \sum_{i=1}^{k} (\tilde{Z}_{i+1} Z_{i+1}^T + \Theta_i K_i^{*T}) \vec{\varphi}_q = 0, \tag{11.6}$$

where $\vec{\varphi}_k,\ k = 1,...,s$ are orthonormal eigenvectors which correspond to the maximal s-multiple eigenvalue λ_1 of the matrix

$$\sum_{i=1}^{k} \left[Z_{i+1} Z_{i+1}^T + K_i^* K_i^{*T} \right],$$

Θ_i are arbitrary matrices which have the same dimension as matrices K_i, and the matrices $\tilde{Z}_{i+1}$ satisfy the equations

$$\tilde{Z}_{i+1} = \tilde{Z}_i - \tilde{Z}_{i+1} A_i + \Theta_i C_i;\ \ \tilde{Z}_{k+1} = 0;\ \ i = 1,...,k.$$

Obviously

$$\begin{aligned}
\sum_{i=1}^{k} \tilde{Z}_{i+1} Z_{i+1}^T &= \sum_{i=1}^{k} \left(\tilde{Z}_{i+1} Z_{i+1}^T + \tilde{Z}_{i+1} S_{i+1} - \tilde{Z}_i S_i \right) \\
&= \sum_{i=1}^{k} \left(\tilde{Z}_{i+1} Z_{i+1}^T + \tilde{Z}_{i+1} \left(S_i + A_i S_i - Z_{i+1}^T \right) - \tilde{Z}_i S_i \right) \\
&= \sum_{i=1}^{k} \left[\left(\tilde{Z}_i - \tilde{Z}_{i+1} A_i + \Theta_i C_i \right) S_i + \tilde{Z}_{i+1} A_i S_i - \tilde{Z}_i S_i \right] \\
&= \sum_{i=1}^{k} \Theta_i C_i S_i.
\end{aligned}$$

Using this equality and the auxiliary systems of equations (11.5) we obtain that (11.6) equals

$$\sum_{q=1}^{s} \vec{\varphi}_q^T p_q \sum_{i=1}^{k} \Theta_i \left(K_i^{*T} + C_i S_i \right) \vec{\varphi}_q = 0.$$

From this equation we obtain all assertions of Theorem 11.1.

Consider the case when $\vec{\xi}_i$ and $\vec{\eta}_i$ are error vectors of dimensions m and n, respectively, which satisfy the inequality

$$\left(\vec{\xi},\vec{\eta}\right) \in G, \;\; G = \left\{ \left(\vec{\xi},\vec{\eta}\right) : \sum_{i=1}^{k} \left(\left\| \vec{\xi}_i \right\|^2 + \left\| \vec{\eta}_i \right\|^2 \right) \le 1 \right\}$$

for some fixed k (where we have put $\vec{\xi} = (\xi_1, \cdots, \xi_k)^T$, and similarly $\vec{\eta}$). Consider the following problem of estimating the state $\vec{x}_k$: to find matrices $\hat{K}_i$ of dimension $m \times n$ and a vector $\vec{l}$ of dimension m which minimize the expression

$$\varphi\left(K_1, \cdots, K_k; \vec{l}\right) = \max_{\left(\vec{\xi},\vec{\eta}\right)\in G} \left\| \vec{x}_{k+1} - \sum_{i=1}^{k} K_i \vec{y}_i - \vec{l} \right\|^2 .$$

The vector

$$\hat{\vec{x}}_{k+1} = \sum_{i=1}^{k} \hat{K}_i \vec{y}_i + \hat{\vec{l}}$$

is called linear minimax estimator of $\vec{x}_{k+1}$.

Repeating the proof of Theorem 10.1 we get

THEOREM 11.2. *Under the above formulated assumptions*

$$\min_{K_i \in R^{m\times n}; \vec{l} \in R^m} \max_{\left(\vec{\xi},\vec{\eta}\right)\in G} \left\| \vec{x}_{k+1} - \sum_{i=1}^{k} K_i \vec{y}_i - \vec{l} \right\|^2 = \lambda_{\max} \left\{ \sum_{i=1}^{k} \left(Z_{i+1} Z_{i+1}^T + K_i^* K_i^{*T} \right) \right\},$$

where the matrices Z_i satisfy the recursive equations

$$Z_{i+1} \left(I + A_i\right) = Z_i + \hat{K}_i C_i; i = 1, ..., k; \;\; Z_{k+1} = I.$$

Moreover, $\hat{\vec{l}} = Z_1 x_1$ and the matrices $\hat{K}_i; i = 1, ..., k$ satisfy the system of equations

$$\sum_{l=1}^{s} p_l \left[\hat{K}_i^T + C_i S_i \right] \vec{\varphi}_l \vec{\varphi}_l^T = 0; \;\; i = 1, ..., k \; , \tag{11.7}$$

where

$$p_l > 0; \;\; l = 1, ..., s, \;\; \sum_{j=1}^{s} p_l = 1,$$

$\vec{\varphi}_k, \;\; k = 1, ..., s$ are orthonormal eigenvectors which correspond to the maximal s-multiple eigenvalue $\lambda_{\max}$ of the matrix

$$\sum_{i=1}^{k} \left[Z_{i+1} Z_{i+1}^T + \hat{K}_i \hat{K}_i^T \right]$$

and the matrices S_i satisfying the system of equations

$$S_{i+1} = S_i + A_i S_i - Z_{i+1}^T; \;\; p = 1, ..., k; \;\; S_1 = 0,$$

one of the solutions of equation (11.7) is

$$\hat{K}_i^T = -C_i S_i, \;\; i = 1, ..., k.$$

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