Linear models with an arbitrary set of perturbations. Spectral equation S_5

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Assume that a linear regression model

$$\vec{y} = X\vec{c} + \vec{\varepsilon}$$

is given, where $\vec{c}$ is an unknown m-dimensional vector, $\vec{y}$ is an n-dimensional vector of observations,$X = (x_{ij}), j = 1, , m, i = 1, , n; n \geq m$ is a matrix, and $\vec{\varepsilon}$ is an n-dimensional vector of unobservable perturbations.

Let the vectors $\vec{c}$ and $\vec{\varepsilon}$ belong to the measurable bounded domains $C \subset K_m$ and $E \subset K_n$, respectively. By means of a linear transformation of the vector $\vec{y} : T_{m\times n}\vec{y} + \vec{t}_m$ we find a matrix $\hat{T}_{m\times n}$ and vector $\hat{\vec{t}}$ which minimize the loss function

$$\varphi\left(T,\ \vec{t}\right) := \sup_{\vec{\varepsilon}\in E,\ \vec{c}\in C} f\left\{\left(T\vec{y} - \vec{t} - \vec{c}\right)^T V\left(T\vec{y} - \vec{t} - \vec{c}\right)\right\},$$

where $V_{m\times m}$ is a nonnegative definite matrix and f is a certain differentiable function which satisfies conditions of Theorem 5.1. The vector $\hat{\vec{c}} = \hat{T}\vec{y} + \hat{\vec{t}}$ is called the S_5-estimator (or minimax estimator) of the vector $\vec{c}$.

Using the main spectral equation as in Section 5 we have that

$$\inf_{T\in L_{m\times n},\vec{t}\in K_m} \sup_{\vec{\varepsilon}\in \mathrm{E},\ \vec{c}\in C} f\left\{\left(T\vec{y} - \vec{t} - \vec{c}\right)^T V\left(T\vec{y} - \vec{t} - \vec{c}\right)\right\}$$
$$= \inf_{T\in L_{m\times n},\vec{t}\in K_m} \lim_{N\to\infty} \lambda_1\left[A_N\left(T,\ \vec{t}\right)\right],$$

where

$$A_N\left(T,\ \vec{t}\right) = (a_{ij})_{i,j=1}^{N},$$

$$a_{ij} = \int\limits_{\vec{\varepsilon}\in E,\ \vec{c}\in C} \theta_i\left(\vec{\varepsilon},\ \vec{c}\right)\theta_j\left(\vec{\varepsilon},\ \vec{c}\right) f\left\{\left[\left(TX - I\right)\vec{c} - T\vec{\varepsilon} - \vec{t}\right]^T V\left[\left(TX - I\right)\vec{c} - T\vec{\varepsilon} - \vec{t}\right]\right\}$$
$$\times \prod_{\substack{p=1,\ldots,n\\ l=1,\ldots m}} \mathrm{d}\varepsilon_p d\,c_l,$$

$\theta_j(\vec{\varepsilon},\ \vec{c})$ is an arbitrary orthonormal system of functions in the domain $C \times E$.

Therefore, we can change our problem: find matrix T_N^* and vector $\vec{t}_N^*$ which minimize the expression

$$\inf_{T\in L_{m\times n},\vec{t}\in K_m} \lambda_1\left[A_N\left(T,\ \vec{t}\right)\right] = \lambda_1\left[A_N\left(T_N^*,\ \vec{t}_N^*\right)\right].$$

Using the proof of Theorem 6.2 we obtain that the matrix $T_N^* \subset L_{m\times n}$ and the vector $\vec{t}_N^* \subset K_m$ satisfy the system of equations S_5.

$$\sum_{k=1}^{s} \vec{\nu}_k^T \left\{ \int_{\vec{\varepsilon}\in \mathrm{E},\ \vec{c}\in C} \theta_i(\vec{\varepsilon},\ \vec{c})\theta_j(\vec{\varepsilon},\ \vec{c}) V\left(\left(T_N^* X - I\right)\vec{c} - T_N^*\vec{\varepsilon} - \vec{t}_N^*\right)\right.$$
$$\times \left(\vec{c}^T X^T - \vec{\varepsilon}^T\right) \frac{\partial f}{\partial u} \left\{\left[\left(T_N^* X - I\right)\vec{c} - T_N^*\vec{\varepsilon} - \vec{t}_N^*\right]^T\right.$$
$$V\left[\left(T_N^* X - I\right)\vec{c} - T_N^*\vec{\varepsilon} - \vec{t}_N^*\right]\right\} \prod_{p,l} \mathrm{d}\varepsilon_p\, \mathrm{d}c_l;$$

$$\sum_{k=1}^{s} \vec{\nu}_k^T \left\{ \int_{\vec{\varepsilon}\in \mathrm{E},\ \vec{c}\in C} \theta_i(\vec{\varepsilon},\ \vec{c})\theta_j(\vec{\varepsilon},\ \vec{c}) V\left(\left(T_N^* X - I\right)\vec{c} - T_N^*\vec{\varepsilon} - \vec{t}_N^*\right)\right\}_{i,j=1}^{N}$$
$$\times \frac{\partial f}{\partial u} \left\{\left[\left(T_N^* X - I\right)\vec{c} - T_N^*\vec{\varepsilon} - \vec{t}_N^*\right]^T V\left[\left(T_N^* X - I\right)\vec{c} - T_N^*\vec{\varepsilon} - \vec{t}_N^*\right]\right\}$$
$$\times \prod_{p,l} \mathrm{d}\varepsilon_p\, \mathrm{d}c_l \vec{v}_k p_k = 0,$$

where $\vec{v}_k,\ \ k = 1, \cdots, s$ are the eigenvectors corresponding to the maximal j-multiple eigenvalue $\lambda_1\left(A_N\right)$ of the matrix A_N,

$$\sum_{k=1}^{s} p_k = 1;\ \ p_k > 0,\ \ k = 1, ..., s.$$

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