# Spectral equation $S_4$ for the estimators of the solutions of systems of Equations with "internal" perturbations in a system of observations

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In this section we develop an approach to the estimation of a solution of a system of equations with random coefficients and random errors in a system of observations.

Let the system of linear equations

$$A\vec{x} = \vec{h} + \vec{\xi}_1,$$

be given, where $A$ is a nondegenerate $n \times n$ matrix, $\vec{x}, \vec{h}, \vec{\xi}_1$ are the vectors of dimension $n$, the matrix A and a vector $\vec{h}$ are known and the value of a vector $\vec{\xi}_1$ is an unknown vector.

Assume that the observed vector $\vec{y}$ of dimension $4m$ is connected with the vector $\vec{x}$ by the equation

$$\vec{y} = \Xi\vec{x} + \vec{\xi}_2$$

where $\Xi$ is a random matrix, $\vec{\xi}_2$ is an unknown vector of dimension $m$, but $\vec{\xi}_1$ and $\vec{\xi}_2$ satisfy the inequality

$$\left\|\vec{\xi}_1\right\|^2 + \left\|\vec{\xi}_2\right\|^2 \leq 1.$$

The problem is to estimate $\vec{x}$ (optimally in a certain sense) by a linear transform of $\vec{y}$. More precisely, we seek for an $n \times n$ matrix $K^*$ and a vector $\vec{l}^*$ of dimension $n$ which minimize

$$\mathbf{E} \max_{\left\|\vec{\xi}_1\right\|^2 + \left\|\vec{\xi}_2\right\|^2 \leq 1} \left\|\vec{x} - K^*\vec{y} - \vec{l}^*\right\|^2.$$

The vector

$$\vec{x}^* = K^*\vec{y} + \vec{l}^*$$

is called a spectral ($S_4$-estimator) or minimax estimator of the vector $\vec{x}$.

Without loss of generality the vector $\vec{h}$ can be chosen to be zero.

Theorem 9.1. *If the matrix $A$ is nondegenerate, then*

$$\min_{K \in L_{n\times m}, \vec{l} \in K_n} \mathbf{E} \max_{\left\|\vec{\xi}_1\right\|^2 + \left\|\vec{\xi}_2\right\|^2 \leq 1} \left\|\vec{x} - K^*\vec{y} - \vec{l}^*\right\|^2$$
$$= \mathbf{E}\,\lambda_1 \left\{ (I - K^*\Xi)\, A^{-1} A^{-1T} (I - K^*\Xi)^T + K^* K^{*T} \right\} \tag{9.1}$$

*and the matrix $K^*$ satisfies the equation*

$$\mathbf{E}\left\{-\Xi A^{-1}A^{-1T}\left(I-K^*\Xi\right)^T+K^{*T}\right\}\sum_{k=1}^{s}p_k\vec{\varphi}_k(D)\vec{\varphi}_k^T(D)=0, \tag{9.2}$$

*where $\vec{\varphi}_k$ are orthonormal eigenvectors corresponding to the $s$-multiple maximal eigenvalue of the random matrix*

$$D=\left(I-K^*\Xi\right)A^{-1}A^{-1T}\left(I-K^*\Xi\right)^T+K^*K^{*T},$$

$\vec{l}^*=0$, *and $p_k>0$ are random variables, such that*

$$\sum_{k=1}^{s}p_k=1.$$

Note that the number $s$ is also a random variable. In particular, when the matrix $\Xi$ has a density, then the maximal eigenvalue is simple with probability 1 and in this case $s=1,\ p_1=1$.

The proof is almost the same as in Section 6. It is sufficient to note that, as in that section, we get

$$\max_{\|\vec{\xi}_1\|^2+\|\vec{\xi}_2\|^2\le 1}\left\|\vec{x}-K^*\vec{y}\right\|^2=\lambda_1\left\{B\left(K\right)B^T\left(K\right)\right\},$$

where $B\left(K\right)=\left[\left(I-K\Xi\right)A^{-1},K\right]$.

In a similar manner we can generalize all previous theorems from Sections 5-8 of this chapter when some matrices in the system of equations are random.

For instance, let us assume that the observed vector $\vec{y}$ of dimension $n$ is connected with the vector $\vec{x}$ by the equation

$$\vec{y}=\Xi\vec{x}+\vec{\xi}_2,$$

where $\Xi$ is a known $n\times m$ matrix, $\vec{\xi}_2$ is an unknown vector of dimension $n$, and $\vec{\xi}_1$ and $\vec{\xi}_2$ satisfy the inequality

$$\left\|\vec{\xi}_1\right\|^2+\left\|\vec{\xi}_2\right\|^2\le 1.$$

The problem is to estimate $\vec{x}$ (optimally in a certain sense) by a linear transform of $\vec{y}$. More precisely, we seek for an $m\times n$ matrix $\hat{K}$ and a vector $\hat{\vec{l}}$ of dimension $m$ which minimize the loss function

$$\varphi\left(K,\vec{l}\right)=\max_{\|\vec{\xi}_1\|^2+\|\vec{\xi}_2\|^2\le 1}\left\|\vec{x}-K\vec{y}-\vec{l}\right\|^2.$$

The vector

$$\hat{\vec{x}} = \hat{K}\vec{y} + \hat{\vec{l}}$$

is called a spectral or *minimax estimator* of the vector $\vec{x}$. Without loss of generality the vector $\vec{h}$ can be chosen to be zero.

THEOREM 9.2. *If the matrix $A$ is regular, then*

$$\min_{K\in R^{m\times n},\vec{l}\in R^m} \max_{\|\vec{\xi}_1\|^2+\|\vec{\xi}_2\|^2\le 1} \left\|\vec{x} - K\vec{y} - \vec{l}\right\|^2$$
$$= \lambda_{max}\left\{\left(I - \hat{K}\Xi\right) A^{-1}A^{-1T}\left(I - \hat{K}\Xi\right)^T + \hat{K}\hat{K}^T\right\},$$

*and the matrix $\hat{K}$ satisfies the equation*

$$\left\{-\Xi A^{-1}A^{-1T}\left(I - \hat{K}\Xi\right)^T + \hat{K}^T\right\}\sum_{k=1}^{s} p_k\vec{\varphi}_k(D)\vec{\varphi}_k^T(D) = 0,$$

*where $\vec{\varphi}_k(D)$ are orthonormal eigenvectors corresponding to the $s$-multiple maximal eigenvalue of the matrix*

$$D = \left(I - \hat{K}\Xi\right) A^{-1}A^{-1T}\left(I - \hat{K}\Xi\right)^T + \hat{K}\hat{K}^T,\ \ \hat{\vec{l}} = 0,$$

*and*

$$\sum_{k=1}^{s} p_k = 1,\ \ p_k > 0,\ \ k = 1, ..., s.$$

The proof is almost the same as in Section 5. It is sufficient to note that, as there, we get

$$\max_{\|\vec{\xi}_1\|^2+\|\vec{\xi}_2\|^2\le 1} \|\vec{x} - K\vec{y}\|^2 = \lambda_{max}\left\{(I - K\Xi)\, A^{-1}A^{-1T}\,(I - K\Xi)^T + KK^T\right\}.$$

Equation (9.2) can be solved numerically by means of various methods of solution of functional equations, for example, by the principle of condensed mappings. Instead of expectation in this equation we can consider a normalized sum of independent realizations of random matrices and use the Monte Carlo method.

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