Linear models with nonrandom perturbations. Spectral equation S_3

V. L. GIRKO

Department of Probability and Statistics, Michigan State University
East Lansing, Michigan 48824

Assume that a linear regression model

$$\vec{y} = X\vec{c} + \vec{\varepsilon}$$

is given, where $\vec{c}$ is an unknown m-dimensional vector, $\vec{y}$ is an n-dimensional vector

$$X = (x_{ij}), \ \ i = 1, ..., n; \ \ j = 1, ..., m; \ n \geq m$$

is a matrix, and is an n-dimensional vector of unobservable perturbations. Let the vectors $\vec{c}$ and $\vec{\varepsilon}$ satisfy the inequality

$$\|\vec{\varepsilon}\|^2 + \beta \|\vec{c}\|^2 \leq 1, \ 0 < \beta < \infty.$$

By means of a linear transformation of the vector $\vec{y}$:

$$T_{m\times n}\vec{y} + \vec{t}_m$$

we find a matrix $\hat{T}_{m\times n}$ which minimizes the loss function

$$\varphi := \max_{\|\vec{\varepsilon}\|^2+\beta\|\vec{c}\|^2\leq 1} (T\vec{y} - \vec{c})^T V (T\vec{y} - \vec{c}),$$

where $V_{m\times m}$ is a nonnegative definite matrix. The vector $\vec{\hat{c}} = \hat{T}\vec{y}$ is called the S_3 - estimator (or minimax estimator) of the vector $\vec{c}$. The motivation of our problems is confirmed by many practical problems. In many of them it is difficult to verify that perturbations are random and also it is difficult to find covariance matrices of random perturbations. Since the perturbations belong to complex sets, we simplify problems considering ellipses which contain these sets. As in Section 5 we obtain that

$$\min_{T\in L_{m\times n}} \max_{\|\vec{\varepsilon}\|^2+\beta\|\vec{c}\|^2\leq 1} (T\vec{y} - \vec{c})^T V (T\vec{y} - \vec{c})$$
$$= \lambda_1 \left\{ \beta^{-1}\sqrt{V} (T^*X - I) (T^*X - I)^T \sqrt{V} + \sqrt{V}T^*{T^*}^T\sqrt{V} \right\},$$

and that the matrix T^* satisfies the spectral equation S_3

$$\left\{ X (T^*X - I)^T \beta^{-1} + {T^*}^T \right\} \sum_{k=1}^{j} \sqrt{V}\vec{v}_k\vec{v}_k^T\sqrt{V}p_k = 0, \tag{8.1}$$

where $\vec{v}_k, \ k = 1, \cdots, j$ are the orthogonal eigenvectors corresponding to the maximal eigenvalue of the matrix

$$\beta^{-1}\sqrt{V}\left(T^{*}X-I\right)\left(T^{*}X-I\right)^{T}\sqrt{V}+\sqrt{V}T^{*}T^{*T}\sqrt{V},\;\; p_k>0,\;\sum\nolimits_{k=1}^{j}p_k=1.$$

One of the solutions of equation (8.1) is

$$T^{*}=X^{T}\left(\beta I+XX^{T}\right)^{-1}.$$

REFERENCES

1. V.L. Girko, S.I. Lyascko and A.G. Nakonechny. On Minimax Regulator for Evolution Equations in Hilbert Space Under Conditions of Uncertainty. *Cybernetics*, N.1, 1987, 67–68 p.
2. V.L. Girko. Spectral Equations S_2 for Minimax Estimations of Solutions of Some Linear Systems*. Third International Workshop on Model Oriented Data Analysis (Moda-3) St. Petersburg, Petrodvorets, Russia 25–30 May 1992 p.11.
3. V.L. Girko. Spectral Equations S_1 and S_2 for Minimax Estimations of Solutions of some Linear Systems Linear Minimax Estimation - Theory and Practice 3–4 August 1992, Oldenburg.
4. V.L. Girko. Spectral Equation for Minimax Estimator of Parameters of Linear Systems*. *Calculating and Applied Mathematics*, N.63, 1987, 114–115 p. Translation in *J. Soviet Math.* **66** 2221–2222 (1993).
5. V.L. Girko and N. Christopeit. Minimax Estimators for Linear Models with Nonrandom Disturbances. *Random Operators and Stochastic Equations* **3**, N.4, 1995, 361–377 p.
6. V.L. Girko. Spectral Theory of Minimax Estimation, *Acta Applicandae Mathematicae* **43**, 1996, 59–69 p.
7. V.L. Girko. **Theory of Linear Algebraic Equations with Random Coefficients**, (monograph), Allerton Press, Inc. New York 1996, 320 p.