Linear Models with Unknown Covariance Matrix. Spectral Equation S_2

V. L. GIRKO

Department of Probability and Statistics, Michigan State University
East Lansing, Michigan 48824

Assume that the unknown m-dimensional vector $\vec{c}$ satisfies the system of equations

$$\vec{y} = X\vec{c} + \vec{\varepsilon},$$

where $\vec{y}$ is an n-dimensional vector of observations of the random vector,

$$X = (x_{ij}), \ \ i = 1, ..., n; \ \ j = 1, ..., m; \ \ n \geq m$$

is a matrix, and $\vec{\varepsilon}$ is an n-dimensional random vector of unobservable perturbations such that

$$\mathbf{E}\,\vec{\varepsilon} = 0, \quad \mathbf{E}\,\vec{\varepsilon}\vec{\varepsilon}^T = R.$$

Let the vector $\vec{c}$ and the matrix R satisfy the inequalities

$$\vec{c}^T D\vec{c} \leq \alpha, \quad \operatorname{Tr} R \leq b,$$

where D is a positive definite matrix,

$$0 < \alpha < \infty, \ 0 < b < \infty.$$

By means of a linear transformation of the vector $\vec{y}$:

$$T_{m\times n}\vec{y} + \vec{t}_m$$

we find a matrix $\hat{T}_{m\times n}$ and a vector $\hat{\vec{t}}_m$ which minimize the loss function

$$f\left(T, \vec{t}\right) = \max_{\vec{c}:\vec{c}^T D\vec{c}\leq a, \vec{c}\in K_m, R\in L_{n\times n}, TrR\leq\beta} E\left(T\vec{y} + \vec{t} \pm \vec{c}\right)^T V \left(T\vec{y} + \vec{t} \pm \vec{c}\right),$$

where $V_{m\times m}$ is a nonnegative definite matrix. The vector

$$\hat{\vec{c}} = \hat{T}_{m\times n}\vec{y} + \hat{\vec{t}}_m$$

is called the S_2-estimator of the vector $\vec{c}$. As in Theorem 6.1 we prove the following assertion.

THEOREM 7.1. *If the matrix D is nondegenerate, then*

$$\min_{\substack{T_{m\times n}\in L_{m\times n},\\ \vec{t}_m\in K_m}} f\left(T, \vec{t}\right) = \alpha\lambda_1 \left[D^{-12}\left(I - \hat{T}_{ij}X\right)^T V \left(I - \hat{T}_{ij}X\right) D^{-12}\right] + \beta\mu_1\left(\hat{T}_{ij}^T V \hat{T}_{ij}\right),$$

where $V^{12}\hat{\vec{t}}_m = 0$, λ_1, μ_1 *are the maximal eigenvalues of multiplicity* i *and* j, *respectively, and the matrices* $\hat{T}_{ij}$ *satisfy the* S_2*-equation*

$$-a\sum_{k=1}^{i} V\left(I-\hat{T}_{ij}X\right)D^{-12}\vec{\varphi}_k\vec{\varphi}_k^T D^{-12}X^T p_k + \beta\sum_{k=1}^{j} d_k V\hat{T}_{ij}\vec{\psi}_k\vec{\psi}_k^T = 0,$$

where

$$p_k, d_k > 0,\ \sum\nolimits_{k=1}^{i} p_k = 1,\ \sum\nolimits_{k=1}^{j} d_k = 1,$$

$$p_k, d_k > 0,\ \sum\nolimits_{k=1}^{i} p_k = 1,\ \sum\nolimits_{k=1}^{j} d_k = 1$$

are orthonormal eigenvectors which corresponding to eigenvalues λ_1 *and* μ_1 *respectively.*

REFERENCES

1. V.L. Girko, S.I. Lyascko and A.G. Nakonechny. On Minimax Regulator for Evolution Equations in Hilbert Space Under Conditions of Uncertainty. *Cybernetics*, N.1, 1987, 67–68 p.
2. V.L. Girko. Spectral Equations S_2 for Minimax Estimations of Solutions of Some Linear Systems*. Third International Workshop on Model Oriented Data Analysis (Moda-3) St. Petersburg, Petrodvorets, Russia 25–30 May 1992 p.11.
3. V.L. Girko. Spectral Equations S_1 and S_2 for Minimax Estimations of Solutions of some Linear Systems Linear Minimax Estimation - Theory and Practice 3–4 August 1992, Oldenburg.
4. V.L. Girko. Spectral Equation for Minimax Estimator of Parameters of Linear Systems*. *Calculating and Applied Mathematics*, N.63, 1987, 114–115 p. Translation in *J. Soviet Math.* **66** 2221–2222 (1993).
5. V.L. Girko and N. Christopeit. Minimax Estimators for Linear Models with Nonrandom Disturbances. *Random Operators and Stochastic Equations* **3**, N.4, 1995, 361–377 p.
6. V.L. Girko. Spectral Theory of Minimax Estimation, *Acta Applicandae Mathematicae* **43**, 1996, 59–69 p.
7. V.L. Girko. **Theory of Linear Algebraic Equations with Random Coefficients**, (monograph), Allerton Press, Inc. New York 1996, 320 p.