## 7. $G_7$-ESTIMATOR OF THE STATES OF DISCRETE CONTROL SYSTEMS

Now we briefly discuss some questions of GSA related to our topic. Increasing demands for the quality of operation of industrial robots led to the necessity of creating better methods of control that take into account dynamic characteristics of manipulators. In order to construct such control systems, it is necessary to have full knowledge of a mathematical model of the manipulator. The dynamic model of the manipulator is a system of nonlinear differential equations. Coefficients of these equations are connected in a rather complicated fashion via trigonometric functions with generalized coordinates of the manipulator. Such a system is complicated for practical use because of the essential nonlinearity and mutual influence of links. Therefore, a simplified mathematical model with adaptive adjustment of the parameters in the control process proves to be expedient.

### 7.1. Adaptive approach to the control of manipulator motion

The standard model was given by linear differential equations of the second order in which the desired characteristics of motion were pointed out. An adaptive regulator in accordance with the standard model "adjusts" control of the manipulator according to the desired motion.

Linearized with respect to the nominal motion, the mathematical model was used in a procedure of control synthesis on the basis of asymptotic linear regulators as well as for constructing autoregressive models, representing displacements in separate links. Parameters of the model are estimated in the process of motion, proceeding from the optimization of some quality criterion.

The dynamics are described by a Lagrange equation of the second kind, which depends on unknown parameters of the manipulator. Locally optimal finitely convergent methods of solving inequalities were used for adaptation algorithms. In [Gir54], a method of adaptive control of the manipulator without full knowledge of the mathematical model is proposed, and its characteristics are studied. The estimation of the parameters of the model is made by observations on the manipulator in the block of adaptation. Using these estimates, a linear regulator optimizing generalized energy is constructed. The estimate of the parameters and the controls is made recurrently. The algorithm proposed is locally optimal.

### 7.2. The discrete analog of the control system

The discrete analog of a mathematical model for the control of manipulator motion can be represented in the form

$$\vec{x}_{n+1} = A(\vec{x}_n)\vec{x}_n + B(\vec{x}_n)\vec{u}_n, \tag{7.1}$$

where $\vec{u}_n$ is the vector of control moments.

We define the trajectory of motion of the manipulator in the form of a sequence of points $\vec{a}_i \in R^{2m}, i = 1, 2, ...,$ through which the manipulator has to pass and approximate the dynamic model of the manipulator by a linear model

$$\vec{x}_{n+1} = A_n\vec{x}_n + B_n\vec{u}_n + \vec{\varepsilon}_{n+1}, \tag{7.2}$$

where $A_n$, $B_n$ are unknown matrices, and $\vec{\varepsilon}_{n+1}$ are errors of modelling. Assume that the matrices $A(\vec{x}(t))$, $B(\vec{x}(t))$ in (3) are constant but unknown. Such assumption will be true for local displacements of the manipulator. Then (7.1) can be written in the form $\vec{x}_{n+1} = A\vec{x}_n + B\vec{u}_n$. We make $n > m$ observations of the manipulator under some fixed controls. From the observations, we construct estimators of the matrices $\hat{A}_n,\ \hat{B}_n$.

Using these estimators, we can find the extrapolated position of the manipulator

$$\vec{x}^e_{n+1} = \hat{A}_n\vec{x}_n + \hat{B}_n\vec{u}_n. \tag{7.3}$$

We choose the control $\vec{u}_n$ to minimize the functional

$$I_n\left(\tilde{\vec{u}}\right) = \inf_{\vec{u}_n}\left\{\left\|\vec{a}_{n+1} - \vec{x}^e_{n+1}\right\|^2 + \delta\left\|\vec{u}_n\right\|^2\right\},\ \delta > 0. \tag{7.4}$$

The observed position of the manipulator under this control will be

$$\vec{x}_{n+1} = \hat{A}_n\vec{x}_n + \hat{B}_n\tilde{\vec{u}}_n + \vec{\varepsilon}_{n+1}.$$

Without loss of generality, we assume that $B$ is a known square matrix which has an inverse. The matrix $A$ will be estimated by the least squares method

$$\hat{A}_n = \sum_{s=1}^{n}\left(\vec{x}_s - B\tilde{\vec{u}}_{s-1}\right)\vec{x}^T_{s-1}\left[\sum_{s=1}^{n}\vec{x}_{s-1}\vec{x}^T_{s-1}\right]^{-1}.$$

Controls from (7.4) will be given in the form ($G_7$–estimator)

$$\tilde{\vec{u}}_s = \left[\delta I + BB^T\right]^{-1}B^T\left(\vec{a}_{s+1} - \tilde{A}_s\vec{x}_s\right),$$

where

$$\tilde{A}_s = \tilde{A}_s\chi\left\{\left\|\tilde{A}_s\right\| < \|A\|\right\} + \tilde{A}_{s-1}\chi\left\{\left\|\tilde{A}_s\right\| \geq \|A\|\right\}$$

and $\chi\left\{\left\|\tilde{A}_s\right\| \geq \|A\|\right\}$ is the indicator of a random event. Given $\tilde{\vec{u}}_n$, we observe the vector $\vec{x}_{n+1}$ again, find $\tilde{\vec{u}}_{n+1}$, , and continue these calculations up to the moment of time $s$ when $\left\|\vec{a}_s - \tilde{\vec{x}}_s\right\|^2 < \varepsilon$, where $\varepsilon > 0$ is a given number. We prove convergence of estimates of the matrix $A$.

### 7.3. The main assertion

THEOREM 7.1. [Gir54, p.518] *Let the following conditions hold:*

$$\mathbf{E}\left\{\vec{\varepsilon}_{n+1}/\vec{\varepsilon}_1,\dots,\vec{\varepsilon}_n\right\} = 0,\ n = 1,2,\dots,$$

$$\sup_n\|\vec{a}_n\| < \infty,$$

$$\|A\|\left\{1 + \left\|\left[\delta I + BB^T\right]^{-1}BB^T\right\|\right\} < 1,$$

$$\sup_n\mathbf{E}\,\|\vec{\varepsilon}_n\|^4 < \infty,$$

$$\sup_n\left\|\left[n^{-1}\sum_{s=1}^{n}\mathbf{E}\,\vec{\varepsilon}_{s-1}\vec{\varepsilon}^T_{s-1}\right]^{-1}\right\| < \infty,$$

$$\limsup_{h\to\infty} \sup_{\substack{|i_p-j_p|\geq h,\\ p=1,\dots,3}} \Big|\mathbf{P}\left\{\varepsilon_{i_p} < x_{i_p}, \varepsilon_{j_p} < x_{i_p}, p=1,...,3\right\}$$
$$-\mathbf{P}\left\{\varepsilon_{i_p} < x_{i_p}, p=1,\dots,3\right\}\mathbf{P}\left\{\varepsilon_{j_p} < x_{i_p}, p=1,\dots,3\right\}\Big| = 0.$$

*Then*

$$\lim_{n\to\infty} \mathbf{E}\left\|\vec{a}_s - \tilde{\vec{x}}_s\right\|^2 \leq c\delta \left\|\left[\delta I + BB^T\right]^{-1}\right\|^2,$$

*and distribution functions of entries of matrix* $\left[\hat{A}_n - A\right] n^{1/2}$ *are asymptotically normal.*

The proposed adaptive method was used for solving some control problem.

### 7.4. $G$-system of recursion equations

We will study estimators of parameters of systems with $m_n$ unknown parameters and with the number $n$ of observations satisfying the $G$-condition:

$$\limsup_{n\to\infty} m_n n^{-1} < \infty.$$

Namely

$$\vec{y}_k = \Theta\vec{y}_{k-1} + \vec{b}_{k-1} + \vec{\varepsilon}_k,$$

where $\Theta = \{\theta_{ij}\}_{i,j=1}^{m_n}$ is an unknown matrix, $\vec{y}_k$, $k = 1, 2, ...$ are $m_n$-dimensional observations, $\vec{y}_0$, $\vec{b}_{k-1}$, $k = 1, 2, ...$ are known vectors, $\vec{\varepsilon}_k$, $k = 1, 2, ...$ are $m_n$-dimensional random vectors. Note, that in the general case, the matrix $\sum_{k=1}^n \vec{y}_{k-1}\vec{y}_{k-1}^T$ can be degenerate. Therefore, we will search for an estimate of a matrix $\Theta = \{\theta_{ij}\}_{i,j=1}^{m_n}$ in regularized form:

$$\hat{\Theta}_n = \sum_{k=1}^n c_n^{-1}\left(\vec{y}_k - \vec{b}_{k-1}\right)\vec{y}_{k-1}^T \left[I_{m_n}\alpha + c_n^{-1}\sum_{k=1}^n \vec{y}_{k-1}\vec{y}_{k-1}^T\right]^{-1},$$

where $\alpha > 0$, and $c_n$ is a certain sequence of numbers. Hence

$$\hat{\Theta}_n = \Theta c_n^{-1} Y_n \left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}$$
$$= \sum_{k=1}^n c_n^{-1}\vec{\varepsilon}_k\vec{y}_{k-1}^T \left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1},$$

where

$$Y_n = \sum_{k=1}^n \vec{y}_{k-1}\vec{y}_{k-1}^T.$$

Let us represent this estimator in the following form

$$n^{-1}\mathrm{Tr}\, Q\left\{\hat{\Theta}_n - \Theta c_n^{-1}Y_n\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}\right\}$$
$$= \mathrm{Tr}\, Q\sum_{k=1}^n c_n^{-1}\vec{\varepsilon}_k\vec{y}_{k-1}^T\left\{n^{-1}\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1} - \mathbf{E}\, n^{-1}\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}\right\}$$
$$+ \mathrm{Tr}\, Q\sum_{k=1}^n c_n^{-1}\vec{\varepsilon}_k\vec{y}_{k-1}^T\left\{\mathbf{E}\, n^{-1}\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}\right\},$$

where $Q=\{q_{ij}\}_{i,j=1}^{m_n}$ is the matrix of real parameters.

### 7.5. Self-averaging of $G$-estimators

Let us find conditions of consistency of the $G$-estimator. We need some auxiliary statements.

LEMMA 7.1. [Gir44, p.220] *If the random vectors* $\vec{\varepsilon}_k,\ k=1,2,...$ *are independent,* $\|\Theta\|<1,\ \mathbf{E}\,\vec{\varepsilon}_k=0,\ k=1,2,...,$ *and*

$$\sup_n \max_{k=1,\dots,n} \mathbf{E}\,\|\vec{\varepsilon}_k\|^2 c_n^{-1}<\infty,$$

*then*

$$\operatorname*{plim}_{n\to\infty}\left\{n^{-1}\mathrm{Tr}\,\left[I_{m_n}\alpha+c_n^{-1}Y_n\right]^{-1}-\mathbf{E}\,n^{-1}\mathrm{Tr}\,\left[I_{m_n}\alpha+c_n^{-1}Y_n\right]^{-1}\right\}=0.$$

LEMMA 7.2. [Gir44, p.220] *If the random vectors* $\vec{\varepsilon}_k,\ k=1,2,...$ *are independent,* $\|\Theta\|<1,\ \mathbf{E}\,\vec{\varepsilon}_k=0,\ k=1,2,...,$ *and*

$$\sup_n \max_{k=1,\dots,n} \mathbf{E}\,\|\vec{\varepsilon}_k\|^2 c_n^{-1}<\infty,$$

*then*

$$\operatorname*{plim}_{n\to\infty}\left\{n^{-1}\mathrm{Tr}\,Qc_n^{-1}Y_n\left[I_{m_n}\alpha+c_n^{-1}Y_n\right]^{-1}-\mathbf{E}\,n^{-1}\mathrm{Tr}\,Qc_n^{-1}Y_n\left[I_{m_n}\alpha+c_n^{-1}Y_n\right]^{-1}\right\}=0.$$

Thus, if the conditions of Lemma 7.2 are satisfied and random variables

$$\left\|\sum_{k=1}^n c_n^{-1}\vec{\varepsilon}_k\vec{y}_{k-1}^T\right\|$$

are bounded in probability, then

$$\begin{aligned}&\frac{1}{n}\mathrm{Tr}\,Q\left\{\hat{\Theta}_n-\Theta\mathbf{E}\,\frac{1}{c_n}Y_n\left[I_{m_n}\alpha+\frac{1}{c_n}Y_n\right]^{-1}\right\}\\ &\cong \mathbf{E}\,\frac{1}{n}\mathrm{Tr}\,Q\sum_{k=1}^n\frac{1}{c_n}\vec{\varepsilon}_k\vec{y}_{k-1}^T\left[I_{m_n}\alpha+\frac{1}{c_n}Y_n\right]^{-1}.\end{aligned}$$

Suppose, the matrix $R=\mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha+c_n^{-1}Y_n\right]^{-1}$ is nondegenerate. Then we consider the $G_7$-estimator

$$\hat{\Theta}_n\left\{\mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha+c_n^{-1}Y_n\right]^{-1}\right\}^{-1}$$

of matrix $\Theta$.

THEOREM 7.2. [Gir44, p.220] *If the random vectors* $\vec{\varepsilon}_k,\ k=1,2,...$ *are independent,* $\|\Theta\|<1,\ \mathbf{E}\,\vec{\varepsilon}_k=0,\ k=1,2,...,$ *and*

$$\sup_n \max_{k=1,\ldots,n} \mathbf{E}\, \|\vec{\varepsilon}_k\|^2 c_n^{-1} < \infty,$$

$$\limsup_{n\to\infty} \left\| \sum_{k=1}^{n} c_n^{-1} \vec{\varepsilon}_k \vec{y}_{k-1}^T \right\| < \infty,$$

$$\limsup_{n\to\infty} \left\| \left\{ \mathbf{E}\, c_n^{-1} Y_n \left[ I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\}^{-1} \right\| < \infty$$

*then*

$$p\lim_{n\to\infty} \left\| \hat{\Theta}_n \left\{ \mathbf{E}\, c_n^{-1} Y_n \left[ I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\}^{-1} - \Theta \right\| = 0.$$

Suppose, the matrix

$$R = \mathbf{E}\, c_n^{-1} Y_n \left[ I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1}$$

is nondegenerate. We consider the $G_7$-estimator

$$\hat{\Theta}_n \left\{ \mathbf{E}\, c_n^{-1} Y_n \left[ I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\}^{-1}$$

of the matrix $\Theta$.

Theorem 7.3. [Gir44, p.220] *If the random vectors $\vec{\varepsilon}_k$, $k = 1, 2, \ldots$ are independent, $\|\Theta\| < 1$, $\mathbf{E}\,\vec{\varepsilon}_k = 0$, $k = 1, 2, \ldots$, and*

$$\sup_n \max_{k=1,\ldots,n} \mathbf{E}\, \|\vec{\varepsilon}_k\|^2 c_n^{-1} < \infty,$$

$$p\limsup_{n\to\infty} \left\| \sum_{k=1}^{n} c_n^{-1} \vec{\varepsilon}_k \vec{y}_{k-1}^T \right\| < \infty,$$

$$\limsup_{n\to\infty} \left\| \left\{ \mathbf{E}\, c_n^{-1} Y_n \left[ I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\}^{-1} \right\| < \infty,$$

*then*

$$p\lim_{n\to\infty} \left\| \hat{\Theta}_n \left\{ \mathbf{E}\, c_n^{-1} Y_n \left[ I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\}^{-1} - \Theta \right\| = 0.$$


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