6. G_6-ESTIMATOR OF STIELTJES' TRANSFORM OF COVARIANCE MATRIX PENCIL

In multivariate analysis, we generally wish to test the following three hypotheses:

I. Equality of the correlation matrices of two n-variate normal populations.

II. Equality of the m-dimensional mean vectors for l-variate normal populations.

III. Independence between m-set and q-set of variates in $(m+q)$ -variate normal population, with $m < q$.

Often the normalized spectral functions of the covariance matrices pencil are used for a verification of these tests.

A large series of papers is devoted to the analysis of normalized spectral functions of the empirical covariance matrices pencil (see reviews and books on the spectral theory of random matrices in the References of this book). However, for many years, nobody could solve the problem of obtaining an equation for Stieltjes' transform of spectral functions of large order empirical covariance matrices when observations of the random vector are independent. In this section, we propose a new G_6-estimator initially presented in [Gir44, Gir54] to solve this problem.

Let the vectors $\vec{x}_1, \ldots, \vec{x}_n$ of dimension m_n be a sample of independent observations of the random vector $\vec{\eta}$, $\mathbf{E}\vec{\eta} = \vec{a}$, and $\mathbf{E}(\vec{\eta}-\vec{a})(\vec{\eta}-\vec{a})^{\mathrm{T}} = R_{m_n}$. Let $\hat{R}_{m_n}$ be the empirical covariance matrix:

$$\hat{R}_{m_n} = n^{-1} \sum_{k=1}^{n} (\vec{x}_k - \hat{\vec{a}})(\vec{x}_k - \hat{\vec{a}})^{\mathrm{T}}, \quad \hat{\vec{a}} = n^{-1} \sum_{k=1}^{n} \vec{x}_k.$$

The statistic

$$\mu_{m_n}(x, R_{m_n}) = m_n^{-1} \sum_{p=1}^{m_n} \chi\{\lambda_p(R_{m_n}) < x\}$$

is called a normalized spectral function of the matrix R_{m_n}. Here, χ is the indicator function and $\lambda_p(R_{m_n})$ are the eigenvalues of the matrix R_{m_n}.
Consider nonsingular covariance matrices R_1 and R_2 of the independent m-dimensional random vectors $\vec{\xi}_1$ and $\vec{\xi}_2$, $\vec{a}_1 = \mathbf{E}\,\vec{\xi}_1$, $\vec{a}_2 = \mathbf{E}\,\vec{\xi}_2$. The statistic

$$\mu_n(x, R_1, R_2) = m^{-1} \sum_{k=1}^{m} \chi\{\lambda_k(R_1, R_2) < x\}$$

is called the normalized spectral function of the covariance matrix R_1 and R_2 pencil. Here $\lambda_k(R_1, R_2)$ are the roots of the characteristic equation

$$\det[R_1 z - R_2] = 0.$$

To avoid confusion, we will assume that the inverse matrix R_1^{-1} exists. Sometimes we will use another definition of the normalized spectral function of the covariance matrices R_1 and R_2 pencil

$$\mu_n(x, R_1, R_2) = m^{-1} \sum_{k=1}^{m} \chi\{\lambda_k(R_1^{-1} R_2) < x\},$$

where $\lambda_k(R_1^{-1}R_2)$ are eigenvalues of matrix $R_1^{-1}R_2$.

Consider Stieltjes' transform with the real parameter

$$\int_0^\infty \frac{\mathrm{d}\mu_n(x, R_1, R_2)}{t+x} = m^{-1}\frac{\partial}{\partial t}\ln\det[R_1 t + R_2]$$
$$= m^{-1}\mathrm{Tr}R_1[R_1 t + R_2]^{-1}, \quad t > 0.$$

Let $\vec{x}_1, \ldots, \vec{x}_{n_1}$ and $\vec{y}_1, \ldots, \vec{y}_{n_2}$ be independent observations of two independent m-dimensional random vectors $\vec{a}_1 + R_1^{1/2}\vec{\xi}_1$ and $\vec{a}_2 + R_2^{1/2}\vec{\xi}_2$,

$$\vec{\xi}_1^{\mathrm{T}} = \{\xi_{11}, \ldots, \xi_{1m}\}, \qquad \vec{\xi}_2^{\mathrm{T}} = \{\xi_{21}, \ldots, \xi_{2m}\}.$$

Let random components $\xi_{11}, \ldots, \xi_{1m}; \quad \xi_{21}, \ldots, \xi_{2m}$ be independent for every m and consider empirical covariance matrices and mean vectors

$$\hat{R}_1 = n_1^{-1}\sum_{k=1}^{n_1}(\vec{x}_k - \hat{\vec{x}})(\vec{x}_k - \hat{\vec{x}})^{\mathrm{T}}, \quad \hat{\vec{x}} = n_1^{-1}\sum_{k=1}^{n_1}\vec{x}_k,$$

$$\hat{R}_2 = n_2^{-1}\sum_{k=1}^{n_2}(\vec{y}_k - \hat{\vec{y}})(\vec{y}_k - \hat{\vec{y}})^{\mathrm{T}}, \quad \hat{\vec{y}} = n_2^{-1}\sum_{k=1}^{n_2}\vec{y}_k.$$

The expression

$$\mu_n(x, \hat{R}_1, \hat{R}_2) = m^{-1}\sum_{k=1}^{\nu}\chi\{\lambda_k(\hat{R}_1, \hat{R}_2) < x\}$$

is called the normalized spectral function of the covariance matrix $\hat{R}_1$ and $\hat{R}_2$ pencil. Here $\lambda_k(\hat{R}_1, \hat{R}_2)$ are the roots of the characteristic equation $\det[\hat{R}_1 z - \hat{R}_2] = 0$ and ν is a discrete random variable. Obviously, if $\hat{R}_1^{-1}$ exists with probability 1, then $\nu = m$ with probability 1.

We study Stieltjes' transform with the real parameter

$$\int_0^\infty \frac{\mathrm{d}\mu_n(x, \hat{R}_1, \hat{R}_2)}{t+x} = m^{-1}\frac{\partial}{\partial t}\ln\det[\hat{R}_1 t + \hat{R}_2]$$
$$= m^{-1}\mathrm{Tr}\,\hat{R}_1[\hat{R}_1 t + \hat{R}_2]^{-1}, \quad t > 0.$$

Let us write this expression as

$$m^{-1}\mathrm{Tr}\,\hat{R}_1[\hat{R}_1 t + \hat{R}_2]^{-1} = -\int_0^\infty \frac{\partial}{\partial t}m^{-1}\mathrm{Tr}[I\alpha + \hat{R}_1 t + \hat{R}_2]^{-1}\,\mathrm{d}\alpha.$$

It can be shown (see [Gir44], [Gir54]]) that under mild conditions on empirical covariance matrices we can consider instead of this integral, the following expression

$$-\frac{\partial}{\partial t}\int_\varepsilon^A m^{-1}\mathrm{Tr}[I\alpha + \hat{R}_1 t + \hat{R}_2]^{-1}\mathrm{d}\alpha + o(\varepsilon) + o(A^{-1}).$$

Here $\varepsilon > 0$ is a small number and A is a large number. Therefore, we can study the covariance matrices pencil with the help of normalized traces of the resolvent of the sum of covariance matrices $\hat{R}_1$ and $\hat{R}_2$:

$$m^{-1}\mathrm{Tr}[I\alpha + \hat{R}_1 t + \hat{R}_2]^{-1}, \quad \alpha > 0, \quad t > 0.$$

Let us consider Stieltjes' transform

$$b(z,\alpha) = \int_0^\infty \frac{\mathrm{d}\mu_{m_n}(x, \hat{R}_1 + \alpha\hat{R}_2)}{x - z} = m_n^{-1}\mathrm{Tr}\big[\hat{R}_1 + \alpha\hat{R}_2 - zI_{m_n}\big]^{-1}, \; z = t + \mathrm{i}s, \; s > 0$$

and the canonical equation for the matrix $C(z) = (c_{pl}(z))_{p,l=1}^{m_n}$

$$C(z,\alpha) = \left\{ n_1^{-1}\sum_{k=1}^{n_1}\mathbf{E}\frac{\vec{\eta}_k\vec{\eta}_k^{\mathrm{T}}}{1 + n_1^{-1}\vec{\eta}_k^{\mathrm{T}}C(z,\alpha)\vec{\eta}_k} + \alpha n_2^{-1}\sum_{k=1}^{n_2}\mathbf{E}\,\frac{\vec{\nu}_k\vec{\nu}_k^T}{1 + n_2^{-1}\vec{\nu}_k^T C(z,\alpha)\vec{\nu}_k} - zI_m \right\}^{-1},$$

where $\vec{\eta}_k = \{\eta_{pk};\, p = 1,\ldots,m\}^{\mathrm{T}} = \vec{x}_k - \vec{a}_1$, $\vec{\nu}_k = \{\nu_{pk};\; p = 1,\ldots,m\}^T = \vec{y}_k - \vec{a}_2$ and I_m is the identity matrix, $s > 0$. In [Gir84] it is shown that under some conditions with probability 1

$$\lim_{n_1,n_2\to\infty}\big[b(z,\alpha) - m^{-1}\mathrm{Tr}\,C(z,\alpha)\big] = 0.$$

Using the proof of Theorem 3.1 we get that under some conditions

$$\begin{aligned} &m^{-1}\mathrm{Tr}\,\hat{R}_1\left[\hat{R}_1 t + \hat{R}_2\right]^{-1} \\ &= \int_0^\infty \frac{\mathrm{d}\mu_n\left(x, R_1, R_2\right)}{\alpha + \frac{t}{1 + tmn_1^{-1}b_m(t,\alpha)} + x\left\{1 + (\alpha - 1)\,mn_2^{-1} + \frac{b_m(t,\alpha)}{1 + tmn_1^{-1}b_m(t,\alpha)}\right\}}, \end{aligned}$$

where

$$b_m\left(t,\alpha\right) = m^{-1}\mathrm{Tr}\,\hat{R}_1\left[\hat{R}_1 t + \hat{R}_2\right]^{-1}.$$

We transform this expression as

$$\begin{aligned} &b_m\left(t,\alpha\right)\left\{1 + (\alpha - 1)\,mn_2^{-1} + \frac{b_m\left(t,\alpha\right)}{1 + tmn_1^{-1}b_m\left(t,\alpha\right)}\right\} \\ &= \int_0^\infty \frac{\mathrm{d}\mu_n\left(x, R_1, R_2\right)}{\left\{1 + (\alpha - 1)\,mn_2^{-1} + \frac{b_m(t,\alpha)}{1 + tmn_1^{-1}b_m(t,\alpha)}\right\}^{-1}\left[\alpha + \frac{t}{1 + tmn_1^{-1}b_m(t,\alpha)}\right] + x}. \end{aligned}$$

Now replace t by the function $\theta\left(t\right)$ which is the nonnegative solution of the equation

$$\left\{1 + (\alpha - 1)\,mn_2^{-1} + \frac{b_m\left(\theta\left(t\right),\alpha\right)}{1 + tmn_1^{-1}b_m\left(\theta\left(t\right),\alpha\right)}\right\}^{-1}\left[\alpha + \frac{\theta\left(t\right)}{1 + tmn_1^{-1}b_m\left(\theta\left(t\right),\alpha\right)}\right] = t.$$

Then we obtain

$$G_6 = b_m\left(\theta\left(t\right), \alpha\right)\left\{1 + \left(\alpha - 1\right) m n_2^{-1} + \frac{b_m\left(\theta\left(t\right), \alpha\right)}{1 + t m n_1^{-1} b_m\left(\theta\left(t\right), \alpha\right)}\right\}.$$

From [Gir44, p.218], [Gir54] we get under $t > 0$

$$p \lim_{n_1, n_2 \to \infty}\left[G_6 - \int_0^{\infty} \frac{\mathrm{d}\mu_n\left(x, R_1, R_2\right)}{t + x}\right] = 0,$$

or

$$p \lim_{n_1, n_2 \to \infty}\left[G_6 - m^{-1} \mathrm{Tr} R_1\left[R_1 t + R_2\right]^{-1}\right] = 0.$$

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