5. G_5-ESTIMATOR OF SMOOTHED NORMALIZED SPECTRAL FUNCTION OF SYMMETRIC MATRICES

Let $\mu_n(x)$ be a normalized spectral function of a covariance matrix R_m. The G_2-estimator for Stieltjes' transform of this function is equal to (see Section 2.2)

$$G_2(A,B,u+\mathrm{i}v)=\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^{A}\operatorname{Im}G_2(z)\,\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(v-\mathrm{i}u)}\mathrm{d}p,\ \ v>0.$$

Using this estimator we can try to find a consistent estimator of $\mu_n(x)$. But in this case two questions arise:

1). Will the estimator G_2 be equal to Stieltjes' transform of a distribution function?

2). The spectral function $\mu_n(x)$ may have a discontinuity. Therefore it is very difficult to use the inverse formula for Stieltjes' transform for finding $\mu_n(x)$, using the G_2-estimator.

To overcome these difficulties we can use the so-called smoothed normalized spectral functions

$$\mu_n(x,\varepsilon)=\frac{1}{\pi}\int_{-\infty}^{x}\operatorname{Im}\frac{1}{m_n}\operatorname{Tr}\left[R_m-(y+\mathrm{i}\varepsilon)\right]^{-1}\mathrm{d}y,\ \ \varepsilon>0.$$

It can be shown that

$$\mu_n(x,\varepsilon)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mu_n(x+\varepsilon y)}{1+y^2}\,\mathrm{d}y,\ \ \varepsilon>0.$$

Therefore we call $\mu_n(x,\varepsilon)$ a smoothed normalized spectral function. Consider the estimator

$$G_5(A,B,x,\varepsilon)=\frac{1}{\pi}\int_{-\infty}^{x}\operatorname{Im}G_2(A,B,y+\mathrm{i}\varepsilon)\,\mathrm{d}y,\ \ \varepsilon>0.$$

It is easy to prove that under the conditions of Theorem 2.1 such estimator G_5 of $\mu_n(x,\varepsilon)$ is consistent: with probability one, for any $\varepsilon>0$ and x

$$\lim_{B\to\infty}\lim_{A\to\infty}\lim_{n\to\infty}\left\{G_5(A,\,B,\,x,\,\varepsilon)-\mu_n(x,\,\varepsilon)\right\}=0.$$