4. CLASS OF G_4 -ESTIMATORS FOR THE TRACES OF THE POWERS OF COVARIANCE MATRICES

We recall that the G_2-estimator is the most important in general statistical analysis. With its help, we can find G_4-estimators of the traces of analytic functions of covariance matrices. Let us show that with the help of the G_2-estimator we can find G_4-estimators of the traces of the powers of covariance matrices. Obviously

$$m_n^{-1}\mathrm{Tr}\, R_{m_n}^k = (-1)^k \, (k!)^{-1} \frac{\partial^k}{\partial t^k} m_n^{-1}\mathrm{Tr}\, [I + tR_{m_n}]_{t=0}^{-1}\,;\;\; k = 1, 2, ...$$

Let us recall too that the G_2-consistent estimator for the traces of resolvents of covariance matrices is found in [Gir39]: $G_2 = m_n^{-1}\mathrm{Tr}\left(I_{m_n} + \hat{\theta}\hat{R}_{m_n}\right)^{-1}$, where $\hat{\theta}$ is the positive solution of the main equation of general statistical analysis

$$\theta\left[1 - m_n n^{-1} + n^{-1}\mathrm{Tr}\left(I + \theta\hat{R}_{m_n}\right)^{-1}\right] = t;\;\; t > 0.$$

Using these estimators after certain simple calculations we find G_4-estimators:
The $G_4^{(1)}$ Estimator of $m_n^{-1}\mathrm{Tr}\, R_{m_n}$ is equal to $m^{-1}\mathrm{Tr}\, \hat{R}_{m_n}$, which is evident. However, to obtain the next estimators of the powers of covariance matrices some calculations are needed.

Theorem 4.1. [Gir44] *The $G_4^{(2)}$ Estimator of $m_n^{-1}\mathrm{Tr}\, R_{m_n}^2$ is equal to*

$$m_n^{-1}\mathrm{Tr}\, \hat{R}_{m_n}^2 - (nm_n)^{-1}\left(\mathrm{Tr}\, \hat{R}_{m_n}\right)^2.$$

4.1. $G_4^{1/2}$-estimator of the square root of covariance matrices

One of the problems of simulation of complex systems is the problem of simulating on computers a normally distributed random vector $\vec{\xi}_m$ with zero mean and given covariance matrix R_{m_n}. Usually one solves such a problem in the following way: first with the help of pseudorandom variables one simulates the standard Normal vector $\vec{\eta}_m$ of dimension m. Then one represents the covariance matrix in the following form:

$$R_{m_n} = T_{m_n}T_{m_n}^T,$$

where T_{m_n} is the upper (or lower) triangular matrix. After this initial preparation we can take pseudorandom vector $\vec{\xi}_m = T_{m_n}\vec{\eta}_m$ or $\vec{\xi}_m = \sqrt{R_{m_n}}\vec{\eta}_m$. Note that the matrix R_{m_n}, as a rule, is unknown. Therefore, we must use a G-estimator of such a matrix. Let us use the integral

$$\sqrt{x} = \frac{2}{\pi}\int_0^\infty \frac{x}{x + t^2}\mathrm{d}t,$$

where $x > 0$ is a real parameter. Similarly, we have for the square root of the covariance matrix

$$R_{m_n}^{1/2} = \frac{2}{\pi}\int_0^\infty R_{m_n}\left\{It^2 + R_{m_n}\right\}^{-1}\mathrm{d}t = \frac{2}{\pi}\int_0^\infty \left\{I - \left[I + t^{-2}R_{m_n}\right]^{-1}\right\}^{-1}\mathrm{d}t.$$

Hence , using the G_2–estimator we can find

$$G_4^{(1/2)} = \frac{2}{\pi} \int_0^{\infty} \left\{ I - \left[I + \hat{\theta}\left(t\right) \hat{R}_{m_n} \right]^{-1} \right\}^{-1} \mathrm{d}t,$$

where $\hat{\theta}\left(t\right)$ is a positive solution of the equation

$$t^2 \theta\left(t\right) \left\{ 1 - \frac{m_n}{n} + \frac{1}{n} \mathrm{Tr} \left[I + \theta\left(t\right) \hat{R}_{m_n} \right]^{-1} \right\} = 1.$$

In [Gir55] it is proven that estimator $G_4^{(1/2)}$ is consistent and asymptotically Normal.

THEOREM 4.3. *If the G-condition* $\limsup_{n\to\infty} m_n n^{-1} < 1$ *is fulfilled, components* $\xi_{ik},\ i = 1, \ldots, m_n$ *of the vectors*

$$\vec{\xi}_k = \left\{\xi_{ik},\ i = 1, \ldots, m_n\right\}^T = R_{m_n}^{-1/2} \left[\vec{x}_k - \vec{a}_k\right],\ k = 1, \ldots, n$$

are independent and for some $\delta > 0$

$$\sup_n \max_{i=1,\ldots,m_n;\, k=1,\ldots,n} \mathbf{E} \left|\xi_{ik}\right|^{4+\delta} < \infty,$$

$$\vec{b}^T \vec{b} < c_1, \quad \vec{a}^T \vec{a} < c_2, \quad 0 < c_3 < \lambda_{\min}\left(R_{m_n}\right) \leq \cdots \leq \lambda_{\max}\left(R_{m_n}\right) \leq c_4,$$

then

$$p\lim_{n\to\infty} \left| \vec{a}^T G_4^{(1/2)} \vec{b} - \vec{a}^T R_{m_n}^{-1/2} \vec{b} \right| = 0.$$

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