21. G_{21}-ESTIMATOR IN THE LIKELIHOOD METHOD

The discussion of this section shows how G-estimators can be constructed from any stochastic experiments. Let $\vec{x}_k,\ k = 1, \ldots, n$ be independent observations of vector $\vec{\xi}$ which has a density $p(\vec{\alpha},\ \vec{x})$, $\vec{x} = \{x_1, \ldots, x_m\}^T$, where $\vec{\alpha} = \{\alpha_1, \ldots, \alpha_l\}^T$ is an unknown vector. The likelihood method consists in the following: as an estimator of vector $\vec{\alpha} = \{\alpha_1, \ldots, \alpha_l\}^T$, we accept any measurable solution $\hat{\vec{\alpha}}$ of the equation

$$\sup_{\vec{\alpha}\in A} L_n(\vec{\alpha}) = L_n\left(\hat{\vec{\alpha}}\right),$$

where

$$L_n(\vec{\alpha}) = \prod_{k=1}^{n} p(\vec{\alpha},\ \vec{x}_k)$$

is the likelihood function. For large dimensional unknown vectors $\vec{\alpha} = \{\alpha_1, \ldots, \alpha_l\}^T$, we consider in this section the general likelihood method or G-Method. In this method we make two assumptions: 1. Instead of estimation of vector $\vec{\alpha} = \{\alpha_1, \ldots, \alpha_l\}^T$ we consider the estimation problem of density $p(\vec{\alpha},\ \vec{x})$, $\vec{x} = \{x_1, \ldots, x_m\}^T$ or a certain functional of this density. This assumption is valid in many important practical problems. II. Instead of one sample of observations $\vec{x}_k,\ k = 1, \ldots, n$ we consider the scheme of series of samples sequence. The number of unknown parameters m and the number of observations n are related so that the following G-condition is fulfilled:

$$\limsup_{n\to\infty} m_n n^{-1} < \infty.$$

Now we give the main form of G-estimators of density $p(\vec{\alpha},\ \vec{x})$:

$$\begin{aligned} G_{21}(\vec{x}) &= \exp\left\{A_\delta - \varepsilon A_\delta^2\right\} p\left(\hat{\vec{\alpha}} + \vec{z},\ \vec{x}\right)_{\vec{z}=0} \\ &= \sum_{k=0}^{\infty} \exp\left\{\lambda_k(\varepsilon) - \delta\lambda_k^2(\varepsilon)\right\} \int p\left(\hat{\vec{\alpha}} + \vec{z},\ \vec{x}\right)\varphi_k(\vec{z})\, \mathrm{d}\vec{z}\varphi_k(\vec{0}), \end{aligned}$$

where $\hat{\vec{\alpha}}$ is the estimator obtained by the likelihood method, $\lambda_k(\varepsilon)$ and $\varphi_{k\varepsilon}(\vec{z})$, $k = 1, 2, \ldots$ denote the eigenvalues and eigenfunctions of the operator $A + \varepsilon q(\vec{z})$, $\vec{z} \in R^m$, respectively

$$A_\varepsilon = \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2}{\partial z_i \partial z_j} \mathbf{E}\left(\hat{\vec{\alpha}} - \vec{\alpha}\right)_i \left(\hat{\vec{\alpha}} - \vec{\alpha}\right)_j + \varepsilon q(\vec{z});\ \varepsilon > 0,\ \delta > 0,$$

and $q(\vec{z})$ is any continuous function satisfying the condition

$$\liminf_{n\to\infty} \lim_{\|\vec{z}\|\to\infty} q(\vec{z}) = \infty.$$

Here we mention some new properties of the G_{21}-estimators we have thus derived:

I. Under the G-condition and some conditions for the density $p(\vec{\alpha}, \vec{x})$, these G_{21}-estimators are consistent and even asymptotically normal.

II. Under the G–condition the standard estimators of the likelihood method lose their asymptotic properties. The G_{21}-estimators of density $p(\vec{\alpha}, \vec{x})$ have a complicated form, but for some cases it can be see that a new estimator of $\vec{\alpha}$ in the expression for the G_{21}–estimator depends on vector $\vec{x} = \{x_1, \dots, x_m\}^T$.

Let us consider one example. Because it is very difficult to find a simple evident expression for the G_{21}-estimator, let

$$p(\vec{\alpha}, \vec{x}) = (2\pi)^{-m_n/2} \exp\left\{-\|\vec{\alpha} - \vec{x}\|^2 /2\right\}, \vec{\alpha} \in R^{m_n}, \vec{x} \in R^p.$$

Then the likelihood estimator of vector $\vec{\alpha}$ is equal to

$$\hat{\vec{\alpha}} = n^{-1} \sum_{k=1}^{n} \vec{x}_k.$$

Putting this estimator in the density $p(\vec{\alpha}, \vec{x})$, we get

$$p\left(\hat{\vec{\alpha}}, \vec{x}\right) = (2\pi)^{-m_n/2} \exp\left\{-\frac{\|\vec{\alpha} - \vec{x}\|^2}{2} + \frac{(\vec{\alpha} - \vec{x})^T \vec{\eta}}{\sqrt{n}} - \frac{\|\vec{\eta}\|^2}{2n}\right\},$$

where $\vec{\eta} = \left(\hat{\vec{\alpha}} - \vec{\alpha}\right)\sqrt{n}$.

Obviously, if

$$\lim_{n\to\infty} m_n n^{-1} = c,\ 0 < c < \infty;\ \limsup_{n\to\infty} \|\vec{\alpha} - \vec{x}\|^2 n^{-1} = 0, \tag{21.1}$$

then

$$\lim_{n\to\infty} \mathbf{E}\left[\frac{p\left(\hat{\vec{\alpha}}, \vec{x}\right)}{p(\vec{\alpha}, \vec{x})} - 1\right]^2 = \left[\mathrm{e}^{-c/2} - 1\right]^2 > 0.$$

But if we take the G_{21}-estimator

$$G\left(\hat{\vec{\alpha}}, \vec{x}\right) = p\left(\hat{\vec{\alpha}}, \vec{x}\right)\mathrm{e}^{c/2}$$

of the density $p(\vec{\alpha}, \vec{x})$, we will have under the same condition (21.1):

$$\lim_{n\to\infty} \mathbf{E}\left[\frac{G_{21}\left(\hat{\vec{\alpha}}, \vec{x}\right)}{p(\vec{\alpha}, \vec{x})} - 1\right]^2 = 0.$$

It is easy to see that the G_{21}-estimator is much better than the standard likelihood estimator.