14. G_{14} -ESTIMATOR OF REGULARIZED DISCRIMINANT FUNCTION

If matrix R is singular or ill-conditioned, then instead of the Mahalanobis distance α, its regularized analog is considered

$$\alpha_\varepsilon = (\vec{a}_1 - \vec{a}_2)^T [\varepsilon I + R]^{-1} (\vec{a}_1 - \vec{a}_2), \ \varepsilon > 0.$$

The regularized distance has more useful properties than the distance α. To prove asymptotic normality of G-estimators, it is not necessary that the random vectors $\vec{\xi}$ and $\vec{\eta}$ be normally distributed. As was mentioned in the previous chapters, the estimator

$$G_\varepsilon = \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right)^T \left[\varepsilon I + \hat{R}\right]^{-1} \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right), \ \varepsilon > 0,$$

with empirical mean vectors and the covariance matrix $\hat{\vec{a}}_1, \ \hat{\vec{a}}_2, \ \hat{R}$, is inappropriate for solving multivariate classification problems. Indeed, with the increase of m, the number of components of the vectors $\vec{\xi}$ and $\vec{\eta}$, the number of observations needed for obtaining a given accuracy in the Mahalanobis distance estimation grows rapidly. In this section we assert that under some conditions, there exists an asymptotically Normal G-estimator for the regularized discriminant function, provided that

$$\limsup_{n_1, n_2 \to \infty} \left[m n_1^{-1} + m n_2^{-1}\right] < \infty.$$

Let $\vec{x}_i, \ \vec{y}_j; \ i = 1, \ldots, n_1; \ j = 1, \ldots, n_2$ be independent observations of m-dimensional independent random vectors $\vec{\xi}$ and $\vec{\eta}$ respectively. We call the expression

$$\begin{aligned} G_{14}(\vec{x}) = &\left\{\vec{x} - \frac{1}{2}\left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right)\right\}^T \left[\varepsilon I + \varepsilon \theta_{n_1,n_2}^{-1} \hat{R}\right]^{-1} \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right) \\ &- \left[n_1^{-1} + n_2^{-1}\right] \varepsilon \theta_{n_1,n_2}^{-1} \mathrm{Tr}\ \hat{R} \left[\varepsilon I + \varepsilon \theta_{n_1,n_2}^{-1} \hat{R}\right]^{-1} \end{aligned}$$

the G_{14}–estimator for the regularized discriminant function. Here θ_{n_1,n_2} is the nonnegative solution of the equation [Gir54, p.615]

$$\begin{aligned} &1 - k_{n_1,n_2} + k_{n_1,n_2} \theta_{n_1,n_2}^{-1} \mathrm{Tr} \hat{R} \left[\theta_{n_1,n_2} I + \hat{R}\right]^{-1} = \varepsilon \theta_{n_1,n_2}, \\ &k_{n_1,n_2} = m \left[n_1 + n_2 - 2\right]^{-1} \end{aligned}$$

It can be seen that there exists a unique nonnegative solution of this equation.

THEOREM 14.1. [Gir54, p.615] *Let the conditions of Theorem 13.1 be satisfied. Then*

$$\begin{aligned} \lim_{n_1,n_2 \to \infty} \max_{i=1,2} \mathbf{P} &\left\{ \left[G_{14}\left(\vec{\xi}_i\right) - \left\{\vec{\xi}_i - \frac{1}{2}(\vec{a}_1 - \vec{a}_2)\right\}^T (\varepsilon I + R)^{-1} (\vec{a}_1 - \vec{a}_2) \right] \right. \\ &\left. \times \sqrt{\frac{n_1 + n_2 - 2}{V_m}} < x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-y^2/2\right\} \mathrm{d}y, \end{aligned}$$

where $\vec{\xi}_i$ is an observation which does not depend on $\vec{x}_i, \ \vec{y}_j; \ i = 1, ..., n_1; \ j = 1, ..., n_2$, distributed as $N\{\vec{a}_1, \ R\}$ or $N\{\vec{a}_2, \ R\}$ and V_m are certain constants.

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