12. G_{12}-REGULARIZED MAHALANOBIS DISTANCE ESTIMATOR

We call

$$\begin{aligned} G_{12} = & \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right)^T \left[\varepsilon I + \varepsilon\theta_{n_1,n_2}^{-1} \hat{R}\right]^{-1} \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right) \\ & - \left[n_1^{-1} + n_2^{-1}\right] \varepsilon\theta_{n_1,n_2}^{-1} \mathrm{Tr}\hat{R} \left[\varepsilon I + \varepsilon\theta_{n_1,n_2}^{-1} \hat{R}\right]^{-1}, \end{aligned} \tag{12.1}$$

the G_{12}-regularized Mahalanobis distance estimator, where $\varepsilon > 0$ is a parameter. Here, θ_{n_1,n_2} is the nonnegative solution of the equation

$$\begin{aligned} & 1 - k_{n_1,n_2} + k_{n_1,n_2}\theta_{n_1,n_2}^{-1} \mathrm{Tr}\hat{R} \left[\theta_{n_1,n_2} I + \hat{R}\right]^{-1} = \varepsilon\theta_{n_1,n_2}, \\ & k_{n_1,n_2} = m \left[n_1 + n_2 - 2\right]^{-1}. \end{aligned}$$

It can be seen that there exists a unique nonnegative solution of this equation.

THEOREM 12.1. [Gir54, p.601] *Let the random variables $\mu_i,\ \nu_i,\ i = 1, \dots, m$ be independent for every n,*

$$\mathbf{E}\,\mu_i = \mathbf{E}\,\nu_i = 0,\ \mathbf{E}\,\mu_i^2 = \mathbf{E}\,\nu_i^2 = 1,\ i = 1, \dots, m,$$

for a certain $\beta > 0$

$$\sup_n \max_{i=1,\dots,m} \mathbf{E} \left[|\mu_i|^{4+\beta} + |\nu_i|^{4+\beta}\right] < \infty$$

$$\inf_n \min_{i=1,\dots,m} \lambda_i (R) > 0, \quad \sup_n \max_{i=1,\dots,m} \lambda_i (R) < \infty,$$

and the G-condition be satisfied:

$$\limsup_{n_1,n_2\to\infty} m \left[n_1 + n_2 - 2\right]^{-1} < \infty, \quad \limsup_{n_1,n_2\to\infty} \left[n_2 n_1^{-1} + n_1 n_2^{-1}\right] < \infty,$$

$$\sup_{n_1,n_2} \left(\vec{a}_1 - \vec{a}_2\right)^T \left(\vec{a}_1 - \vec{a}_2\right) < \infty,$$

then, as $\varepsilon > 0$

$$\begin{aligned} & \lim_{n_1,n_2\to\infty} \mathbf{P} \left\{ \left[G_{12} - \left(\vec{a}_1 - \vec{a}_2\right)^T \left(\varepsilon I + R\right)^{-1} \left(\vec{a}_1 - \vec{a}_2\right)\right] D_m^{-1/2} \sqrt{n_1 + n_2 - 2} < x \right\} \\ & = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-y^2/2\right\} \mathrm{d}y, \end{aligned}$$

where D_m are certain constants.