11. G_{11}-ESTIMATOR OF THE MAHALANOBIS DISTANCE

Let $\vec{x}_i,\ \vec{y}_j;\ i=1,...,n_1;\ j=1,...,n_2$ be independent observations of m-dimensional random vectors $\vec{\xi}=\vec{a}_1+\sqrt{R}\vec{\mu},\ \vec{\mu}^T=\{\mu_i,\ i=1,\dots,m\}$ and $\vec{\eta}=\vec{a}_2+\sqrt{R}\vec{\nu},\ \vec{\nu}^T=\{\nu_i,\ i=1,\dots,m\}$ respectively and suppose that random variables $\mu_i,\ \nu_i,\ i=1,\dots,m$ are independent for every n. As the empirical mean value vectors and the covariance matrix R, we take:

$$\hat{\vec{a}}_1=n_1^{-1}\sum_{i=1}^{n_1}\vec{x}_i,\ \hat{\vec{a}}_2=n_2^{-1}\sum_{i=1}^{n_2}\vec{y}_i,$$

$$\hat{R}=\frac{1}{n_1+n_2-2}\left\{\sum_{i=1}^{n_1}\left(\vec{x}_i-\hat{\vec{a}}_1\right)\left(\vec{x}_i-\hat{\vec{a}}_1\right)^T+\sum_{i=1}^{n_2}\left(\vec{y}_i-\hat{\vec{a}}_2\right)\left(\vec{y}_i-\hat{\vec{a}}_2\right)^T\right\}.$$

We shall refer to the expression

$$G_{11}=\left\{\left(\hat{\vec{a}}_1-\hat{\vec{a}}_2\right)^T\hat{R}^{-1}\left(\hat{\vec{a}}_1-\hat{\vec{a}}_2\right)\right\}\frac{n_1+n_2-2-m}{n_1+n_2-2}-\frac{m}{n_1}-\frac{m}{n_2}$$

as the G_{11}-estimator of the Mahalanobis Distance.

THEOREM 11.1. [Gir54, p.598] *Let* $n_1+n_2-2>m$,

$$\mathbf{E}\,\mu_i=\mathbf{E}\,\nu_i=0,\ \mathbf{E}\,\mu_i^2=\mathbf{E}\,\nu_i^2=1,\ i=1,\dots,m,$$

and for a certain $\beta>0$

$$\sup_n\max_{i=1,\dots,m}\mathbf{E}\left[|\mu_i|^{4+\beta}+|\nu_i|^{4+\beta}\right]<\infty$$

$$\inf_n\min_{i=1,\dots,m}\lambda_i\left(R\right)>0,\ \sup_n\max_{i=1,\dots,m}\lambda_i\left(R\right)<\infty,$$

$$\lim_{n_1,n_2\to\infty}mn_1^{-1}=c_1,\quad \lim_{n_1,n_2\to\infty}mn_2^{-1}=c_2,\quad c_1,c_2<\infty;\quad c_1+c_2\neq c_1c_2,$$

$$\lim_{n_1,n_2\to\infty}\left(\vec{a}_1-\vec{a}_2\right)^TR^{-1}\left(\vec{a}_1-\vec{a}_2\right)\left[n_1^{-1}+n_2^{-1}\right]=0.$$

Then

$$p\lim_{n_1,n_2\to\infty}\left\{G_{11}-\left(\vec{a}_1-\vec{a}_2\right)^TR^{-1}\left(\vec{a}_1-\vec{a}_2\right)\right\}=0.$$

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