10. G_{10}-ESTIMATOR OF THE SOLUTION OF A REGULARIZED DISCRETE KOLMOGOROV-WIENER FILTER WITH KNOWN FREE VECTOR

The discrete analog of a regularized Kolmogorov-Wiener filter has the form

$$(\varepsilon I_m + R_m)\,\vec{\varphi}(t) = \vec{b}(t),$$

where $\varepsilon > 0$ is a parameter of regularization,

$$R_m = \left\{m^{-1} R\left(sm^{-1}, km^{-1}\right)\right\}_{k,s=1}^{m};\ \ \vec{b}^T(t) = \left\{Q\left(t, sm^{-1}\right),\ \ s = 1, \ldots, m\right\},$$

$$\vec{\varphi}^T(t) = \left\{\varphi\left(t, km^{-1}\right),\ \ k = 1, ..., m\right\},$$

$$R(x, y) = \mathbf{E}\left[\alpha(x) - \mathbf{E}\,\alpha(x)\right]\left[\alpha(y) - \mathbf{E}\,\alpha(y)\right],$$

$$Q(x, y) = \mathbf{E}\left[\alpha(x) - \mathbf{E}\,\alpha(x)\right]\left[\beta(y) - \mathbf{E}\,\beta(y)\right].$$

Here $\alpha(x),\ \beta(y)$ are random processes. The estimator $\hat{\vec{\varphi}}(t) = \left(\varepsilon I + \hat{R}_m\right)^{-1} \hat{\vec{b}}(t)$ converges in probability to $\vec{\varphi}(t)$ when $n_1,\ n_2 \to \infty$. Here

$$\hat{R} = \left\{m^{-1}\hat{R}\left(sm^{-1}, km^{-1}\right)\right\}_{k,s=1}^{m},\ \ \vec{\varphi}^T(t) = \left\{\varphi\left(t, km^{-1}\right),\ \ k = 1, ..., m\right\};$$

$$\hat{\vec{b}}^T(t) = \left\{\hat{Q}\left(t, sm^{-1}\right),\ \ s = 1, \ldots, m\right\},$$

$$\hat{R}(x, y) = (n_1 - 1)^{-1}\sum_{k=1}^{n_1}\left[\alpha_k(x) - \hat{\alpha}(x)\right]\left[\alpha_k(y) - \hat{\alpha}(y)\right],$$

$$\hat{Q}(x, y) = (n_2 - 1)^{-1}\sum_{k=1}^{n_2}\left[\alpha_k(x) - \hat{\alpha}(x)\right]\left[\beta_k(y) - \hat{\beta}(y)\right],$$

and $\alpha_k(x),\ \beta_k(y)$ are independent observations of $\alpha(x),\ \beta(y)$. Applying the G-analysis technique, which is described in [Gir44, Gir54, Gir69, Gir84], we can obtain an estimator of $\vec{\varphi}(t)$, which approaches in probability $\vec{\varphi}(t)$, provided that

$$\lim_{n_1\to\infty} mn_1^{-1} < 1;\quad \lim_{n_1\to\infty} mn_2^{-1} < \infty$$

This estimator will be referred to as the G_{10}-estimator:

$$\vec{G}_{10} = \varepsilon^{-1}\left(I + \hat{\theta}\hat{R}_m\right)^{-1}\vec{b}(t), \tag{10.1}$$

where $\hat{\theta}$ is a nonnegative solution of the equation

$$\theta\left[1 - \gamma_{n_1} + \gamma_{n_1} m^{-1}\mathrm{Tr}\ \left(\theta I + \hat{R}_m\right)^{-1}\right] = \varepsilon^{-1},\ \ \varepsilon > 0;\ \ \gamma_{n_1} = mn_1^{-1} < 1. \tag{10.2}$$

THEOREM 10.1. [Gir44, Gir54, Gir69, Gir84] *Assume that*

$$\vec{x}_k := \left\{\alpha_k\left(sm^{-1}\right);\ \ s=1,...,m\right\}^T = R_m^{1/2}\vec{\eta}_k + \vec{a},$$

$$\left\{\vec{\eta}_k^T = \{\eta_{ik};\ i=1,...,m\}\right\};\ k=1,...,n$$

$R_m^{1/2}$ *is a symmetric matrix,* t *is fixed,* $n_1 = n_2 = n$*, random variables* $\eta_{ik};\ i = 1,...,m;\ k=1,...,n$ *are independent for every* n*, and*

$$\mathbf{E}\,\eta_{ik}=0;\quad \mathbf{E}\,\eta_{ik}^2=1;\ \ i=1,\ldots,m;\quad k=1,\ldots,n$$

$\lim\limits_{n\to\infty} mn^{-1}<1,\ \lambda_i(R)\le c<\infty$, *the vector* $\vec{b}$ *is known,*

$$\sup_m\left[\vec{b}^T\vec{b}+\vec{c}^T\vec{c}\right]<\infty,\ \varepsilon>0,$$

where $\vec{c}\in R^m,\ \lambda_i(R)$ *are the eigenvalues of the matrix* R_m*. Then*

$$p\lim_{n\to\infty}\left[\vec{c}^T G_{10}-\vec{c}^T\vec{\varphi}\right]=0.$$

10.1. G_{10}-estimator for the solution of a Kolmogorov-Wiener filter with unknown vector

Consider the discrete analog of a regularized Kolmogorov-Wiener filter

$$\vec{b}(t)=\left(\varepsilon I+R_m\right)\vec{\varphi}(t), \tag{10.3}$$

where $\varepsilon>0$ is a parameter of regularization,

$$R_m=\left\{m^{-1}R\left(sm^{-1},km^{-1}\right)\right\}_{k,s=1}^m;\ \ \vec{b}^T(t)=\left\{Q\left(t,sm^{-1}\right),\ \ s=1,...,m\right\},$$

$$\vec{\varphi}^T(t)=\left\{\varphi\left(t,km^{-1}\right),\ \ k=1,...,m\right\},$$

$$R(x,y)=\mathbf{E}\left[\alpha(x)-\mathbf{E}\,\alpha(x)\right]\left[\alpha(y)-\mathbf{E}\,\alpha(y)\right],$$

$$Q(x,y)=\mathbf{E}\left[\alpha(x)-\mathbf{E}\,\alpha(x)\right]\left[\beta(y)-\mathbf{E}\,\beta(y)\right].$$

For this case, when free vector $\vec{b}(t)$ is unknown, the estimator vector $\vec{\varphi}^T(t)=\{\varphi\left(t,km^{-1}\right),\ k=1,\ldots,m\}$ will be referred to as the $\tilde{\vec{G}}_{10}$ -estimator. It has the form

$$\tilde{\vec{G}}_{10}=\varepsilon^{-1}\left\{1+\varepsilon\hat{\theta}\left[\gamma_n-n^{-1}\mathrm{Tr}\left\{I+\hat{\theta}\hat{R}_m\right\}^{-1}\right]\right\}\left(I+\hat{\theta}\hat{R}_m\right)^{-1}\hat{\vec{b}}, \tag{10.4}$$

where $\hat{\theta}$ is a nonnegative solution of the equation

$$\theta\left[1-\gamma_n+\gamma_n m^{-1}\mathrm{Tr}\left(\theta I+\hat{R}_m\right)^{-1}\right]=\varepsilon^{-1},\;\; \varepsilon>0;\;\; \gamma_n=mn^{-1}<1, \qquad (10.5)$$

$$\hat{R}_m=n^{-1}\sum_{k=1}^{n}R_m^{1/2}\vec{\eta}_k\vec{\eta}_k R_m^{1/2}-\left(\hat{\vec{x}}-\vec{a}\right)\left(\hat{\vec{x}}-\vec{a}\right)^T;\; \hat{\vec{x}}=n^{-1}\sum_{k=1}^{n}\vec{x}_k,$$

$$\hat{\vec{b}}=n^{-1}\sum_{k=1}^{n}\left(y_k-\hat{y}\right)\left(\vec{x}_k-\hat{\vec{x}}\right);\; \hat{y}=n^{-1}\sum_{k=1}^{n}y_k,$$

$$\vec{x}_k:=\left\{\alpha_k\left(sm^{-1}\right);\;\; s=1,...,m\right\}^T=R_m^{1/2}\vec{\eta}_k+\vec{a};\; y_k:=\beta_k\left(t\right)=\xi_k+p,$$

$R_m^{1/2}$ is a symmetric matrix, t is fixed, $n_1=n_2=n$, the vectors

$$\left\{\vec{\eta}_k^T=\left\{\eta_{ik};\; i=1,...,m\right\};\;\; \xi_k\right\};\; k=1,...,n$$

are independent for every n, random variables $\eta_{ik};\; i=1,...,m$ are independent; $\xi_k;\; k=1,...,n$ are also independent, and

$$\mathbf{E}\,\eta_{ik}=0;\; \mathbf{E}\,\eta_{ik}^2=1;\;\; \mathbf{E}\;\xi_k=0;\; \mathbf{E}\,\xi_k\left(\sqrt{R_m}\vec{\eta}_k\right)_{ik}=b_i;\; i=1,...,m;\quad k=1,...,n.$$

THEOREM 10.3. [Gir84, p.298] *If*

$$\limsup_{n\to\infty} mn^{-1}<1,$$

$$\lambda_i\left(R_m\right)\leq c<\infty;\; i=1,...,m$$

$$\sup_m\left[\vec{b}^T\vec{b}+\vec{c}^T\vec{c}\right]<\infty,\; \varepsilon>0$$

$$\sup_n\max_{i=1,...,m;\; k=1,...,n}\mathbf{E}\;\left|\eta_{ik}\right|^4<\infty,$$

$$\sup_n\max_{k=1,...,n}\max_{s=1,...,m}\lambda_s\left\{\mathbf{E}\;\left[\xi_k\sqrt{R_m}\vec{\eta}_k-\vec{b}\right]\left[\xi_k\sqrt{R_m}\vec{\eta}_k-\vec{b}\right]^T\right\}<\infty,$$

then for every $\varepsilon>0$

$$p\lim_{n\to\infty}\left[\vec{c}^T\tilde{\vec{G}}_{10}-\vec{c}^T\vec{\varphi}\right]=p\lim_{n\to\infty}\left[\vec{c}^T\tilde{\vec{G}}_{10}-\vec{c}^T\left(I\varepsilon+R_m\right)^{-1}\vec{b}\right]=0.$$

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