CHAPTER 14

Ten years of General Statistical Analysis. (The main G-estimators of General Statistical Analysis)

The first estimators of General Statistical Analysis appeared ten years ago in [Gir39]. In this analysis we try to find new estimators under two main assumptions. Firstly, we do not require the existence of the density of observations; e.g. we do not require the observations to have a Normal distribution. Secondly, we develop this analysis when the number of parameters can increase together with the number of observations or the observation vector's dimension is comparable in magnitude with the sample size. It is further assumed that the dimension of the parameter space involved in the model remains constant with respect to n even when the dimension m_n of the random vectors increases. These three assumptions have a great significance. Do not confuse this analysis with "Generalized Statistical Analysis " or "Global Statistical Analysis", etc. based on additional information on distribution of observations. General statistical analysis has greatly influenced physics, especially nuclear physics, control theory, linear stochastic programming and so on. Indeed, this analysis inspired the physicist Eugene Wigner to develop the so-called random matrix physics. Random matrix physics has developed so deeply now that we see the inverse process: random matrix physics begins to enrich the General Statistical Analysis. Therefore, we include certain results obtained by physicists who are working in random matrix physics and in dynamical systems with random errors. Let us point out the principal procedures for General Statistical Analysis (GSA). As has been observed in many publications, the large order of a system or the large dimension of observed vectors requires a large sample size. For this reason one needs the most accurate estimators. In mathematical statistics, most results are relevant only when the dimension is fairly small. For large dimensions, common techniques are inefficient. Hence, the study of high dimensional problems is important.

In many applications of statistics, e.g. econometrics, environmental statistics, ecological statistics, taxonomy, biostatistics, etc. investigators often encounter data sets where the number of measured characteristics is large and the size of available data is also substantial. As a result, the problem is very complex theoretically, as well as computationally, because of the many parameters a feature which is sometimes called the "curse of dimensionality". Hence, there is a need to study rigorously methods which will give efficient results under such circumstances. The standard methods of statistical analysis usually require large amounts of computer time, and cannot be recommended for use with large data sets or a large number of parameters. In many applications the number of parameters to be estimated increases indefinitely with the sample size and therefore the estimators are not "consistent" [Mar], [Wald].

Multivariate statistical analysis took a new turn when distributions of observations and their dimensions started to be treated as arbitrary distribution functions and numbers respectively. In applied problems, it is very difficult to verify whether the observed random vectors have either a Normal distribution or an elliptically contoured distribution. Even under the assumption of such distribution, the joint densities of eigenvalues and of the corresponding eigenvectors of empirical covariance matrices are complicated. In fact they involve the Haar measure on the orthogonal group, the matrix hypergeometric function, etc.

A new General Statistical Analysis is developed for these problems. We study a complex system S, such that the number of parameters of the corresponding model can increase together with the number of observations of this system. The importance of this theory lies in showing how by using the observations on system S, one can construct mathematical models (G-estimators) which in some sense approximate the system S at a given rate. In this analysis the existence of the densities of the observed random vectors and matrices is not needed, and the assumptions about the nature of observations are quite general. However, the existence of several moments for their components is required. We make the following assumptions (axioms) and introduce some technical language that is appropriate for all of the following analysis.

AXIOM 1. A SEQUENCE OF RUNNING MODELS M_n OF A SYSTEM S IS GIVEN

We assume that the dimension m_n of the model M_n of a system S can increase together with the number n of observations of a system S. Analysing many practical problems we can confirm that indeed n depends on m_n and cannot grow arbitrarily fast as m_n itself increases. It is supposed there is a sample of observations $x_1, x_2, \dots, x_n$ of a system S. For theoretical analysis of models we consider the sequence of observations $x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}, \quad n = 1, 2, \dots$ of systems S (random arrays). We assume that the dimension m of theoretical vector-observations can change, when the number of observations itself increases, i.e. we assume that we have a sequence of models $M_1, M_2, \dots$. We call this sequence the running models of system S.

AXIOM 2. THE DIMENSION OF AN ESTIMATED FUNCTIONAL $\varphi(S)$ OF A SYSTEM S IS FIXED

In GSA we do not estimate system S, because we apply this analysis when we have a number of observations which is almost the same as the number of unknown parameters. From the analysis of many statistical problems we can conclude that instead of estimating the system S, we must estimate some functional $\varphi(S)$. Therefore, in this analysis, we assume that the dimension (the number of unknown parameters) of the estimated characteristics $\varphi(S)$ of the system S will not change, when the number m_n of parameters of the models M_n of the system S increases. This assumption is met in many practical problems.

AXIOM 3. THE G-CONDITION (THE UNCERTAINTY PRINCIPLE) IS GIVEN AND THE EXISTENCE OF THE "CRITICAL POINT" IS ASSUMED

The numbers of unknown parameters m_n of running models and the number of observations n of system S satisfy the G-condition:

$$\limsup_{n \to \infty} f(m_n, n) \leq \hbar < \infty,$$

where $f(m_n, n)$ is some positive function increasing in m_n and decreasing in n. In most cases $f(x, y)$ can be chosen to be $f(m_n, n) = m_n n^{-1}$. The constant $\hbar$ depends on the system S and is called the "critical point". This means that if

$$\limsup_{n\to\infty} f(m_n, n) > \hbar$$

then it is impossible to find a consistent estimator of a certain functional $\varphi(S)$ of system S.

AXIOM 4. THE SEQUENCE OF PROBABILITY SPACES IS GIVEN. THE PRINCIPLE OF RUNNING PROBABILITY SPACES $(\Omega_n, \mathcal{F}_n, \mathbf{P}_n)$.

In the abstract theory of probability we require the existence of a unique probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and in the corresponding statistical theory of von Mises we require the existence of a limit of empirical probability measures $\hat{\mathbf{P}}_n$, so that in some sense $\lim_{n\to\infty} \hat{\mathbf{P}}_n = \mathbf{P}$. In GSA we replace such condition with the condition where instead of one abstract probability space we have a sequence of certain abstract probability spaces $(\Omega, \mathcal{F}, \mathbf{P}_m), m = 1, 2, ...$ The corresponding empirical measures $\hat{\mathbf{P}}_n(m)$ do not converge in general, although some functional (such as expectation $\vec{a} = \int \vec{x}\hat{\mathbf{P}}_n(m)\,(\mathrm{d}\vec{x})$ or the covariance matrices $\int(\vec{x}-\vec{a})(\vec{x}-\vec{a})^T\hat{\mathbf{P}}_n(m)\,(\mathrm{d}\vec{x})$ of random vectors) converges to the same vector and matrix for the corresponding measure $\mathbf{P}_m$ of the sequence of probability spaces. It is obvious that in this case we have wider application of our theory.

AXIOM 5. A CERTAIN QUALITY CHARACTERISTIC EXISTS

The most important aim in our theory is to define a quality characteristic of the sequence of models $M_n(\omega)$, which themselves differ in corresponding quality characteristics in the strong theoretical analysis and which thereby allows us to consider smoothness quality. We consider the following quality characteristic for $M_m(\omega)$-models

$$I(S,\hbar) = \lim_{n,m\to\infty, nm^{-1}\to\hbar} \sup_{\mathbf{P}_n : \int_\Omega \phi(S,\omega)\,\mathrm{d}\mathbf{P}_n(\omega) = \text{const}} \int_\Omega \|\varphi(S) - \varphi(M_m(\omega)\|\,\mathrm{d}\mathbf{P}_n(\omega),$$

where $\|\cdot\|$ is a distance between the system S and the model $M_n(\omega)$, $\mathbf{P}_n$ is a sequence of probability measures, $\phi(S_n,\omega)$ is a functional.

AXIOM 6. FEEDBACK CONTROL ALSO EXISTS

If the criterion quality characteristic $I(S,\hbar)$ exceeds a certain constant, which we call the "confidential constant" then we have to reach one of two conclusions: 1). Our probability measure is wrong. Then we can try to change $\mathbf{P}_m$ by $\hat{\mathbf{P}}_n(m)$, an empirical measure. 2). Our model M_m is wrong. Then we have to find a new, more precise, model M_{m+1} and calculate new quality characteristic $I(S,\hbar)$ choosing measure $\mathbf{P}_m$ and model M_{m+1}. Therefore, we have to include the feedback control $C(S - M_m)$ in our analysis.

Representing the axioms symbolically we say that GSA is specified if the following seven objects are given

$$\Big\{\hbar,\, S,\, \varphi(S),\, \Omega,\, \mathcal{F},\, \mathbf{P}_n,\, I(S,\hbar), C(S-M_m)\Big\}.$$

In the following sections we present a collection of the main estimators of G-analysis. For some of them it is proven that under certain conditions they are consistent and sometimes even asymptotically Normal. Note that these estimators can significantly decrease the number of observations required to solve many practical problems.

1. G_1-ESTIMATOR OF GENERALIZED VARIANCE

Let the independent observations $\vec{x}_1, ..., \vec{x}_n$ of the m_n-dimensional random vector $\vec{\xi}$, $n > m_n$ be given,

$$\hat{R} := (n-1)^{-1} \sum_{k=1}^{n} (\vec{x}_k - \hat{\vec{x}})(\vec{x}_k - \hat{\vec{x}})^{\mathrm{T}}, \quad \hat{\vec{x}} = n^{-1} \sum_{k=1}^{n} \vec{x}_k.$$

The expression $\det R$ is called a generalized variance. If the vectors $\vec{x}_i,\ i = 1, \ldots, n$ are independent and distributed according to the multidimensional Normal law $N(\vec{a}, R)$, then

$$\det \hat{R} \approx [\det R](n-1)^{-m} \prod_{i=n-m}^{n-1} \chi_i^2,$$

where χ_i^2 are independent random variables with the χ^2-distribution and i degrees of freedom. In the general case, the distribution of $\det \hat{R}$ is intractable, and therefore finding a consistent estimator for $\det R$ is a very complicated problem. It is proved (see [Gir39–41], [Gir43–45], [Gir53–55], [Gir 69], [Gir75]), that under certain conditions the G-estimators for $c_n^{-1} \ln \det R$ equal

$$G_1(\hat{R}) := c_n^{-1} \{\ln \det \hat{R} + \ln[(n-1)^m (A_{n-1}^m)^{-1} n(n-m_n)^{-1}\},$$

where $A_{n-1}^m = (n-1)...(n-m)$, c_n is a sequence of constants such that

$$\lim_{n\to\infty} c_n^{-2} \ln[n(n-m_n)^{-1}] = 0.$$

For every value $n > m_n$, let the m_n-dimensional random vectors $\vec{x}_1^{(n)}, ..., \vec{x}_n^{(n)}$ be independent and identically distributed with a mean vector $\vec{a}$ and nondegenerate covariance matrices R_{m_n}. For a certain $\delta > 0$

$$\sup_n \sup_{\substack{i=1,\ldots,n \\ j=1,\ldots,m_n}} \mathbf{E}\, |\tilde{x}_{ij}^{(n)}|^{4+\delta} < \infty,$$

where $\tilde{\vec{x}}_{ij}^{(n)}$ are the components of the vector $\tilde{\vec{x}}_i = R_{m_n}^{-1/2}(x_i^{(n)} - \vec{a})$, and

$$\lim_{n\to\infty} (n - m_n) = \infty, \quad \lim_{n\to\infty} n m_n^{-1} \geq 1;$$

and for each value of $n > m_n$, let the random variables $\tilde{x}_{ij}^{(n)},\ i = 1, ..., n,\ j = 1, ..., m_n$ be independent. Then (see [Gir39–41], [Gir43–45], [Gir53–55], [Gir 69], [Gir75])

$$p\lim_{n\to\infty}[G_1(\hat{R}_{m_n}) - c_n^{-1} \ln \det R_{m_n}] = 0,$$

$$\lim_{n\to\infty} \mathbf{P}\{[c_n G_1(\hat{R}_{m_n}) - \ln \det R_{m_n}][-2\ln(1 - m_n n^{-1})]^{-1/2} < x\}$$
$$= (2\pi)^{-1/2} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\, \mathrm{d}y.$$

2. G_2-ESTIMATOR OF THE REAL STIELTJES TRANSFORM OF THE NORMALIZED SPECTRAL FUNCTION OF COVARIANCE MATRICES

Consider the main problem of the statistical analysis of observations of large dimension: the estimation of Stieltjes' transforms of the normalized spectral functions

$$\mu_{m_n}(x) = m_n^{-1} \sum_{k=1}^{m_n} \chi(\lambda_k < x)$$

of the covariance matrices R_{m_n} from the observations of the random vector $\vec{\xi}$ with the covariance matrix R_{m_n}, where λ_k are eigenvalues of matrix R_{m_n}. Note that many analytic functions of the covariance matrices that are used in multivariate statistical analysis can be expressed through the spectral function $\mu_{m_n}(x)$. For example, the function

$$m_n^{-1}\mathrm{Tr} f(R_{m_n}) = \int_0^\infty f(x)\,\mathrm{d}\mu_{m_n}(x),$$

where $f(x)$ is an analytical function.

The function

$$\varphi(t, R_{m_n}) = \int_0^\infty (1+tx)^{-1}\,\mathrm{d}\mu_{m_n} = m_n^{-1}\mathrm{Tr}(I + tR_{m_n})^{-1},\ t > 0,$$

is called Stieltjes' transform of the function $\mu_{m_n}(x)$. A consistent estimator of Stieltjes' transform $\varphi(t, R_{m_n})$ is equal to: $G_2(t, \hat{R}_{m_n}) = \varphi(\hat{\theta}_n(t), \hat{R}_{m_n})$, where $\hat{\theta}_n(t)$ is the positive solution of the equation

$$\theta(1 - m_n(n-1)^{-1} + m_n(n-1)^{-1}\varphi(\theta, \hat{R}_{m_n})) = t,\ t \geq 0.$$

It is obvious that the positive solution of this equation exists and is unique as $t \geq 0,\ m_n(n-1)^{-1} < 1$.

Let the independent observations $\vec{x}_1, ..., \vec{x}_n$ of the m_n-dimensional random vector $\vec{\xi}$ be given. Assume that the G-condition is fulfilled:

$$\limsup_{n\to\infty} m_n n^{-1} < 1, \quad 0 < c_1 \leq \lambda_i \leq c_2 < \infty,\ i = 1, ..., m_n,$$

and let the components of the vector $(\eta_{1k}, ..., \eta_{m_n k})^{\mathrm{T}} = R_{m_n}^{-1/2}(\vec{\xi} - \mathbf{E}\vec{\xi})$ be independent, and

$$\sup_n \sup_{k=1,...,n} \sup_{i=1,...,m_n} \mathbf{E}|\eta_{ik}|^{4+\delta} < \infty, \quad \delta > 0.$$

Then ([Gir38–41], [Gir43–45], [Gir54], [Gir58], [Gir69], [Gir76], [Gir84])

$$\lim_{n\to\infty} \mathbf{P}\left\{[G_2(t, \hat{R}_{m_n}) - \varphi(t, R_{m_n})]\sqrt{(n-1)m_n}\, a_n(t) + c_n(t) < x\right\}$$
$$= (2\pi)^{-1/2} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\mathrm{d}y,$$

as $t > 0$, where $a_n(t)$ and $c_n(t)$ are some bounded functions.

2.1. G_2-estimator of a complex Stieltjes transform of the normalized spectral function of covariance matrices

Here, the $G_2(z)$-consistent estimator for the trace of the resolvent of covariance matrices (Stieltjes' transform)

$$m_n^{-1}\mathrm{Tr}\left(\hat{R}_{m_n}-zI_{m_n}\right)^{-1},\ z=t+\mathrm{i}s,\ s>0$$

is given as

$$G_2(z)=z^{-1}\hat{\theta}(z)\,m_n^{-1}\mathrm{Tr}\left\{\hat{R}_{m_n}-\hat{\theta}(z)\,I_{m_n}\right\}^{-1},$$

where $\hat{\theta}(z)$ is the measurable complex solution of the equation

$$\hat{\theta}(z)\,\frac{1}{n}\mathrm{Tr}\left\{\hat{R}_{m_n}-\hat{\theta}(z)\,I_{m_n}\right\}^{-1}-\left(1-\frac{m_n}{n}\right)+\frac{\hat{\theta}(z)}{z}=0.$$

THEOREM 2.1. [Gir45] *Suppose that $\vec{x}_1,\dots,\vec{x}_n$ is a random vector sample,*

$$\vec{x}_k=R_{m_n}^{1/2}\vec{\xi}_k+\vec{a},\ \ \mathbf{E}\,\vec{\xi}_k=0,\ \mathbf{E}\,\vec{\xi}_k\vec{\xi}_k^{\,T}=I_{m_n},\ \ \vec{\xi}_k^{\,T}=\{\xi_{ik},i=1,...,m_n\},$$

for any positive defined matrix A_m whose eigenvalues are bounded by a certain constant

$$\lim_{n\to\infty}\max_{k=1,\dots,n}n^{-1}\mathbf{E}\left|(\vec{x}_k-\vec{a})^T A\,(\vec{x}_k-\vec{a})-n^{-1}\mathrm{Tr}\,R_{m_n}A\right|=0,$$

$$\lambda_i(R_{m_n})<c_1<\infty,\ \ i=1,...,m_n,$$

$$\liminf_{n\to\infty}m_n n^{-1}>0,\ \ \limsup_{n\to\infty}m_n n^{-1}<\infty.$$

Then with probability one for every $S>0$ and $T>0$

$$\lim_{n\to\infty}\sup_{\substack{0<c\le\mathrm{Im}z\le S,\\|\mathrm{Re}z|\le T}}\left|G_2(z)-m_n^{-1}\mathrm{Tr}\left\{R_{m_n}-zI_{m_n}\right\}^{-1}\right|=0,$$

for some $c>0$.

2.2. Modified G_2-estimator

Thus, under some conditions,

$$\lim_{n\to\infty}\sup_{\substack{0<c\le\mathrm{Im}z\le S,\\|\mathrm{Re}z|\le T}}\left|G_2(z)-m_n^{-1}\mathrm{Tr}\left\{R_{m_n}-zI_{m_n}\right\}^{-1}\right|=0,$$

where $c>0$ is a certain constant (which usually is not small). However, we need to know the trace of the resolvent of the covariance matrix for all $s>0$. Since function $m_n^{-1}\mathrm{Tr}\left\{R_{m_n}-zI_{m_n}\right\}^{-1}$ is analytical in z, $\mathrm{Im}z>0$, we can use many methods for its analytical continuation. For example we can use the Fourier transform and consider the following modified G_2 estimator:

$$G_2(A,B,u+\mathrm{i}v)=\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^{A}\mathrm{Im}G_2(z)\,\mathrm{e}^{-itp}\mathrm{d}t\right\}\mathrm{e}^{-p(v-\mathrm{i}u)}\mathrm{d}p,\ \ v>0,$$

where $s > c > 0$.

It is easy to prove that the following assertion is valid:

THEOREM 2.2 [Gir45] *If the conditions of Theorem 2.1 are fulfilled, then with probability one, for every $\varepsilon > 0$,*

$$\lim_{B\to\infty}\lim_{A\to\infty}\lim_{n\to\infty}\sup_{v,\,0<\varepsilon\le u}\left|G_2\left(A,B,u+\mathrm{i}v\right)-m_n^{-1}\mathrm{Tr}\left\{R_{m_n}-\left(u+\mathrm{i}v\right)I_{m_n}\right\}^{-1}\right|=0.$$

2.3. G_2-estimator for the trace of the resolvent of empirical covariance matrix when Lindeberg's condition is not fulfilled

Let $\vec{x}_1,\dots,\vec{x}_n$ be the sample of independent observations of a random vector,

$$\vec{x}_k=R_{m_n}^{1/2}\vec{\xi}_k+\vec{a},\ \ \mathbf{E}\,\vec{\xi}_k=0,\ \mathbf{E}\,\vec{\xi}_k\vec{\xi}_k^T=I_{m_n},\ \ \vec{\xi}_k^T=\left\{\beta_k\xi_{ik},i=1,...,m_n\right\},$$

where β_k are independent and do not depend on variables ξ_{ik}.

For this case, the G-equation for the trace of resolvent has the following form [Gir69]

$$b(z)=\mathbf{E}\,m_n^{-1}\mathrm{Tr}\left[\hat{R}_{m_n}-zI_{m_n}\right]^{-1}=m_n^{-1}\sum_{p=1}^{m_n}\frac{1}{\lambda_p n^{-1}\sum\limits_{i=1}^{n}\mathbf{E}\,\frac{\beta_i^2}{\gamma\beta_i^2q(z)+1}-z}+\varepsilon_n,$$

where $\lim_{n\to\infty}\varepsilon_n=0$ and $q(z)$ is satisfies the equation

$$q(z)=m_n^{-1}\sum_{p=1}^{m_n}\frac{\lambda_p}{\lambda_p n^{-1}\sum\limits_{i=1}^{n}\mathbf{E}\,\frac{\beta_i^2}{\gamma\beta_i^2q(z)+1}-z},\ \ z=t+\mathrm{i}s,\ \ \gamma=\frac{m_n}{n}.$$

Let us express function $q(z)$ through function $b(z)$. One has

$$\begin{aligned}q(z)n^{-1}\sum_{i=1}^{n}\mathbf{E}\,\frac{\beta_i^2}{\gamma\beta_i^2q\left(z\right)+1}&=m_n^{-1}\sum_{p=1}^{m_n}\frac{\lambda_p n^{-1}\sum\limits_{i=1}^{n}\mathbf{E}\,\frac{\beta_i^2}{\gamma\beta_i^2q(z)+1}}{\lambda_p n^{-1}\sum\limits_{i=1}^{n}\mathbf{E}\,\frac{\beta_i^2}{\gamma\beta_i^2q(z)+1}-z}\\&=1+m_n^{-1}\sum_{p=1}^{m_n}\frac{z}{\lambda_p n^{-1}\sum\limits_{i=1}^{n}\mathbf{E}\,\frac{\beta_i^2}{\gamma\beta_i^2q(z)+1}-z}\\&=1+zb\left(z\right).\end{aligned}$$

Hence, in this case, the G_2- estimator has the following form

$$G_2(z)=\hat{b}_n(\theta(z)z)\theta(z),$$

where $\theta(z)$ is any measurable solution of the equation

$$n^{-1}\sum_{i=1}^{n}\mathbf{E}\,\frac{\beta_i^2}{\gamma\beta_i^2q\left(\theta(z)z\right)+1}=\theta(z)$$

and $q(z)$ is any measurable solution of the equation

$$q(z)n^{-1}\sum_{i=1}^{n}\mathbf{E}\,\frac{\beta_i^2}{\gamma\beta_i^2 q\left(z\right)+1}=1+z\hat{b}_n\left(z\right),$$

with

$$\hat{b}_n\left(z\right)=n^{-1}\mathrm{Tr}\left[\hat{R}_{m_n}-Iz\right]^{-1}.$$

3. G_3-ESTIMATOR OF INVERSE COVARIANCE MATRIX

The G_3-estimator of a matrix $R_{m_n}^{-1}$ is equal to

$$G_3=\hat{R}_{m_n}^{-1}\left[1-m_n n^{-1}\right].$$

THEOREM 3.1. ([Gir44], [Gir54]) *If G-condition* $\limsup_{n\to\infty} m_n n^{-1} < 1$ *is fulfilled, components* $\xi_{ik},\ i=1,\ldots,m_n$ *of the vectors*

$$\vec{\xi}_k=\left\{\xi_{ik},\ i=1,\ldots,m_n\right\}^T=R_{m_n}^{-1/2}\left[\vec{x}_k-\vec{a}_k\right],\ k=1,\ldots,n$$

are independent and for some $\delta>0$

$$\sup_n \max_{\substack{i=1,\ldots,m_n;\\ k=1,\ldots,n}}\mathbf{E}\left|\xi_{ik}\right|^{4+\delta}<\infty,$$

$$\vec{b}^T\vec{b}<c_1,\quad \vec{a}^T\vec{a}<c_2,\quad 0<c_3<\lambda_{\min}\left(R_{m_n}\right)\le\cdots\le\lambda_{\max}\left(R_{m_n}\right)\le c_4,$$

then

$$\underset{n\to\infty}{p\lim}\left[\vec{a}^T G_3\vec{b}-\vec{a}^T R_{m_n}^{-1}\vec{b}\right]=0.$$

THEOREM 3.2. ([Gir44], [Gir54]) *If G-condition* $\limsup_{n\to\infty} m_n n^{-1} < 1$ *holds, components* ξ_{ik}, $i=1,\ldots,m_n$ *of the vectors*

$$\vec{\xi}_k=\left\{\xi_{ik},\ i=1,\ldots,m_n\right\}^T=R_{m_n}^{-1/2}\left[\vec{x}_k-\vec{a}_k\right],\ k=1,\ldots,n$$

are independent, have the standard Normal distribution and

$$\vec{b}^T\vec{b}<c_1,\quad ,\vec{a}^T\vec{a}<c_2,\quad \lambda_{\min}\left[R_{m_n}\right]>c_3>0,$$

then

$$\lim_{n\to\infty}\mathbf{P}\left\{\left[\vec{a}^T G_3\vec{b}-\vec{a}^T R_{m_n}^{-1}\vec{b}\right]c_n<x\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left\{-y^2/2\right\}\mathrm{d}y,$$

where c_n *is a certain sequence of constant.*

4. CLASS OF G_4 -ESTIMATORS FOR THE TRACES OF THE POWERS OF COVARIANCE MATRICES

We recall that the G_2-estimator is the most important in general statistical analysis. With its help, we can find G_4-estimators of the traces of analytic functions of covariance matrices. Let us show that with the help of the G_2-estimator we can find G_4-estimators of the traces of the powers of covariance matrices. Obviously

$$m_n^{-1}\mathrm{Tr}\, R_{m_n}^k = (-1)^k\,(k!)^{-1}\,\frac{\partial^k}{\partial t^k} m_n^{-1}\mathrm{Tr}\,[I + tR_{m_n}]_{t=0}^{-1}\,;\;\; k = 1, 2, ...$$

Let us recall too that the G_2-consistent estimator for the traces of resolvents of covariance matrices is found in [Gir39]: $G_2 = m_n^{-1}\mathrm{Tr}\left(I_{m_n} + \hat{\theta}\hat{R}_{m_n}\right)^{-1}$, where $\hat{\theta}$ is the positive solution of the main equation of general statistical analysis

$$\theta\left[1 - m_n n^{-1} + n^{-1}\mathrm{Tr}\left(I + \theta\hat{R}_{m_n}\right)^{-1}\right] = t;\;\; t > 0.$$

Using these estimators after certain simple calculations we find G_4-estimators:
The $G_4^{(1)}$ Estimator of $m_n^{-1}\mathrm{Tr}\, R_{m_n}$ is equal to $m^{-1}\mathrm{Tr}\, \hat{R}_{m_n}$, which is evident. However, to obtain the next estimators of the powers of covariance matrices some calculations are needed.

THEOREM 4.1. [Gir44] *The $G_4^{(2)}$ Estimator of $m_n^{-1}\mathrm{Tr}\, R_{m_n}^2$ is equal to*

$$m_n^{-1}\mathrm{Tr}\,\hat{R}_{m_n}^2 - (nm_n)^{-1}\left(\mathrm{Tr}\,\hat{R}_{m_n}\right)^2.$$

4.1. $G_4^{1/2}$-estimator of the square root of covariance matrices

One of the problems of simulation of complex systems is the problem of simulating on computers a normally distributed random vector $\vec{\xi}_m$ with zero mean and given covariance matrix R_{m_n}. Usually one solves such a problem in the following way: first with the help of pseudorandom variables one simulates the standard Normal vector $\vec{\eta}_m$ of dimension m. Then one represents the covariance matrix in the following form:

$$R_{m_n} = T_{m_n}T_{m_n}^T,$$

where T_{m_n} is the upper (or lower) triangular matrix. After this initial preparation we can take pseudorandom vector $\vec{\xi}_m = T_{m_n}\vec{\eta}_m$ or $\vec{\xi}_m = \sqrt{R_{m_n}}\vec{\eta}_m$. Note that the matrix R_{m_n}, as a rule, is unknown. Therefore, we must use a G-estimator of such a matrix. Let us use the integral

$$\sqrt{x} = \frac{2}{\pi}\int_0^\infty \frac{x}{x + t^2}\mathrm{d}t,$$

where $x > 0$ is a real parameter. Similarly, we have for the square root of the covariance matrix

$$R_{m_n}^{1/2} = \frac{2}{\pi}\int_0^\infty R_{m_n}\left\{It^2 + R_{m_n}\right\}^{-1}\mathrm{d}t = \frac{2}{\pi}\int_0^\infty \left\{I - \left[I + t^{-2}R_{m_n}\right]^{-1}\right\}^{-1}\mathrm{d}t.$$

Hence , using the G_2–estimator we can find

$$G_4^{(1/2)} = \frac{2}{\pi}\int_0^\infty \left\{ I - \left[I + \hat{\theta}\left(t\right)\hat{R}_{m_n}\right]^{-1}\right\}^{-1} \mathrm{d}t,$$

where $\hat{\theta}\left(t\right)$ is a positive solution of the equation

$$t^2\theta\left(t\right)\left\{1 - \frac{m_n}{n} + \frac{1}{n}\mathrm{Tr}\left[I + \theta\left(t\right)\hat{R}_{m_n}\right]^{-1}\right\} = 1.$$

In [Gir55] it is proven that estimator $G_4^{(1/2)}$ is consistent and asymptotically Normal.

THEOREM 4.3. *If the G-condition* $\limsup_{n\to\infty} m_n n^{-1} < 1$ *is fulfilled, components* $\xi_{ik},\ i = 1,\ldots,m_n$ *of the vectors*

$$\vec{\xi}_k = \left\{\xi_{ik},\ i = 1,\ldots,m_n\right\}^T = R_{m_n}^{-1/2}\left[\vec{x}_k - \vec{a}_k\right],\ k = 1,\ldots,n$$

are independent and for some $\delta > 0$

$$\sup_n \max_{i=1,\ldots,m_n;\, k=1,\ldots,n} \mathbf{E}\left|\xi_{ik}\right|^{4+\delta} < \infty,$$

$$\vec{b}^T\vec{b} < c_1,\quad \vec{a}^T\vec{a} < c_2,\quad 0 < c_3 < \lambda_{\min}\left(R_{m_n}\right) \le \cdots \le \lambda_{\max}\left(R_{m_n}\right) \le c_4,$$

then

$$p\lim_{n\to\infty}\left|\vec{a}^T G_4^{(1/2)}\vec{b} - \vec{a}^T R_{m_n}^{-1/2}\vec{b}\right| = 0.$$

5. G_5-ESTIMATOR OF SMOOTHED NORMALIZED SPECTRAL FUNCTION OF SYMMETRIC MATRICES

Let $\mu_n\left(x\right)$ be a normalized spectral function of a covariance matrix R_m. The G_2-estimator for Stieltjes' transform of this function is equal to (see Section 2.2)

$$G_2\left(A,B,u+\mathrm{i}v\right) = \mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^{A}\mathrm{Im}\, G_2\left(z\right)\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(v-\mathrm{i}u)}\mathrm{d}p,\ v > 0.$$

Using this estimator we can try to find a consistent estimator of $\mu_n\left(x\right)$. But in this case two questions arise:

1). Will the estimator G_2 be equal to Stieltjes' transform of a distribution function?

2). The spectral function $\mu_n(x)$ may have a discontinuity. Therefore it is very difficult to use the inverse formula for Stieltjes' transform for finding $\mu_n(x)$, using the G_2-estimator.

To overcome these difficulties we can use the so-called smoothed normalized spectral functions

$$\mu_n\left(x,\varepsilon\right) = \frac{1}{\pi}\int_{-\infty}^{x}\mathrm{Im}\,\frac{1}{m_n}\mathrm{Tr}\left[R_m - \left(y + \mathrm{i}\varepsilon\right)\right]^{-1}\mathrm{d}y,\ \varepsilon > 0.$$

It can be shown that

$$\mu_n(x,\varepsilon) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mu_n(x+\varepsilon y)}{1+y^2}\,\mathrm{d}y,\ \ \varepsilon > 0.$$

Therefore we call $\mu_n(x,\varepsilon)$ a smoothed normalized spectral function. Consider the estimator

$$G_5(A,B,x,\varepsilon) = \frac{1}{\pi}\int_{-\infty}^{x} \operatorname{Im} G_2(A,B,y+\mathrm{i}\varepsilon)\,\mathrm{d}y,\ \ \varepsilon > 0.$$

It is easy to prove that under the conditions of Theorem 2.1 such estimator G_5 of $\mu_n(x,\varepsilon)$ is consistent: with probability one, for any $\varepsilon > 0$ and x

$$\lim_{B\to\infty}\lim_{A\to\infty}\lim_{n\to\infty}\{G_5(A,\,B,\,x,\,\varepsilon) - \mu_n(x,\,\varepsilon)\} = 0.$$

6. G_6-ESTIMATOR OF STIELTJES' TRANSFORM OF COVARIANCE MATRIX PENCIL

In multivariate analysis, we generally wish to test the following three hypotheses:

I. Equality of the correlation matrices of two n-variate normal populations.

II. Equality of the m-dimensional mean vectors for l-variate normal populations.

III. Independence between m-set and q-set of variates in $(m+q)$ -variate normal population, with $m < q$.

Often the normalized spectral functions of the covariance matrices pencil are used for a verification of these tests.

A large series of papers is devoted to the analysis of normalized spectral functions of the empirical covariance matrices pencil (see reviews and books on the spectral theory of random matrices in the References of this book). However, for many years, nobody could solve the problem of obtaining an equation for Stieltjes' transform of spectral functions of large order empirical covariance matrices when observations of the random vector are independent. In this section, we propose a new G_6-estimator initially presented in [Gir44, Gir54] to solve this problem.

Let the vectors $\vec{x}_1,\ldots,\vec{x}_n$ of dimension m_n be a sample of independent observations of the random vector $\vec{\eta}$, $\mathbf{E}\vec{\eta} = \vec{a}$, and $\mathbf{E}(\vec{\eta}-\vec{a})(\vec{\eta}-\vec{a})^{\mathrm{T}} = R_{m_n}$. Let $\hat{R}_{m_n}$ be the empirical covariance matrix:

$$\hat{R}_{m_n} = n^{-1}\sum_{k=1}^{n}(\vec{x}_k - \hat{\vec{a}})(\vec{x}_k - \hat{\vec{a}})^{\mathrm{T}}, \quad \hat{\vec{a}} = n^{-1}\sum_{k=1}^{n}\vec{x}_k.$$

The statistic

$$\mu_{m_n}(x, R_{m_n}) = m_n^{-1}\sum_{p=1}^{m_n}\chi\{\lambda_p(R_{m_n}) < x\}$$

is called a normalized spectral function of the matrix R_{m_n}. Here, χ is the indicator function and $\lambda_p(R_{m_n})$ are the eigenvalues of the matrix R_{m_n}.

Consider nonsingular covariance matrices R_1 and R_2 of the independent m-dimensional random vectors $\vec{\xi}_1$ and $\vec{\xi}_2$, $\vec{a}_1 = \mathbf{E}\,\vec{\xi}_1$, $\vec{a}_2 = \mathbf{E}\,\vec{\xi}_2$. The statistic

$$\mu_n(x, R_1, R_2) = m^{-1}\sum_{k=1}^{m}\chi\{\lambda_k(R_1,R_2) < x\}$$

is called the normalized spectral function of the covariance matrix R_1 and R_2 pencil. Here $\lambda_k(R_1, R_2)$ are the roots of the characteristic equation

$$\det[R_1 z - R_2] = 0.$$

To avoid confusion, we will assume that the inverse matrix R_1^{-1} exists. Sometimes we will use another definition of the normalized spectral function of the covariance matrices R_1 and R_2 pencil

$$\mu_n(x, R_1, R_2) = m^{-1} \sum_{k=1}^{m} \chi\big\{\lambda_k(R_1^{-1} R_2) < x\big\},$$

where $\lambda_k(R_1^{-1}R_2)$ are eigenvalues of matrix $R_1^{-1}R_2$.

Consider Stieltjes' transform with the real parameter

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}\mu_n(x, R_1, R_2)}{t + x} &= m^{-1} \frac{\partial}{\partial t} \ln \det[R_1 t + R_2] \\ &= m^{-1} \mathrm{Tr} R_1 [R_1 t + R_2]^{-1}, \quad t > 0.\end{aligned}$$

Let $\vec{x}_1, \ldots, \vec{x}_{n_1}$ and $\vec{y}_1, \ldots, \vec{y}_{n_2}$ be independent observations of two independent m-dimensional random vectors $\vec{a}_1 + R_1^{1/2}\vec{\xi}_1$ and $\vec{a}_2 + R_2^{1/2}\vec{\xi}_2$,

$$\vec{\xi}_1^{\mathrm{T}} = \{\xi_{11}, \ldots, \xi_{1m}\}, \qquad \vec{\xi}_2^{\mathrm{T}} = \{\xi_{21}, \ldots, \xi_{2m}\}.$$

Let random components $\xi_{11}, \ldots, \xi_{1m}; \quad \xi_{21}, \ldots, \xi_{2m}$ be independent for every m and consider empirical covariance matrices and mean vectors

$$\hat{R}_1 = n_1^{-1} \sum_{k=1}^{n_1} (\vec{x}_k - \hat{\vec{x}})(\vec{x}_k - \hat{\vec{x}})^{\mathrm{T}}, \quad \hat{\vec{x}} = n_1^{-1} \sum_{k=1}^{n_1} \vec{x}_k,$$

$$\hat{R}_2 = n_2^{-1} \sum_{k=1}^{n_2} (\vec{y}_k - \hat{\vec{y}})(\vec{y}_k - \hat{\vec{y}})^{\mathrm{T}}, \quad \hat{\vec{y}} = n_2^{-1} \sum_{k=1}^{n_2} \vec{y}_k.$$

The expression

$$\mu_n(x, \hat{R}_1, \hat{R}_2) = m^{-1} \sum_{k=1}^{\nu} \chi\big\{\lambda_k(\hat{R}_1, \hat{R}_2) < x\big)$$

is called the normalized spectral function of the covariance matrix $\hat{R}_1$ and $\hat{R}_2$ pencil. Here $\lambda_k(\hat{R}_1, \hat{R}_2)$ are the roots of the characteristic equation $\det[\hat{R}_1 z - \hat{R}_2] = 0$ and ν is a discrete random variable. Obviously, if $\hat{R}_1^{-1}$ exists with probability 1, then $\nu = m$ with probability 1.

We study Stieltjes' transform with the real parameter

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}\mu_n(x, \hat{R}_1, \hat{R}_2)}{t + x} &= m^{-1} \frac{\partial}{\partial t} \ln \det[\hat{R}_1 t + \hat{R}_2] \\ &= m^{-1} \mathrm{Tr}\, \hat{R}_1 [\hat{R}_1 t + \hat{R}_2]^{-1}, \quad t > 0.\end{aligned}$$

Let us write this expression as

$$m^{-1}\mathrm{Tr}\,\hat{R}_1[\hat{R}_1 t+\hat{R}_2]^{-1}=-\int_0^\infty \frac{\partial}{\partial t}m^{-1}\mathrm{Tr}[I\alpha+\hat{R}_1 t+\hat{R}_2]^{-1}\,\mathrm{d}\alpha.$$

It can be shown (see [Gir44], [Gir54]]) that under mild conditions on empirical covariance matrices we can consider instead of this integral, the following expression

$$-\frac{\partial}{\partial t}\int_\varepsilon^A m^{-1}\mathrm{Tr}[I\alpha+\hat{R}_1 t+\hat{R}_2]^{-1}\mathrm{d}\alpha+o(\varepsilon)+o(A^{-1}).$$

Here $\varepsilon>0$ is a small number and A is a large number. Therefore, we can study the covariance matrices pencil with the help of normalized traces of the resolvent of the sum of covariance matrices $\hat{R}_1$ and $\hat{R}_2$:

$$m^{-1}\mathrm{Tr}[I\alpha+\hat{R}_1 t+\hat{R}_2]^{-1},\quad \alpha>0,\quad t>0.$$

Let us consider Stieltjes' transform

$$b(z,\alpha)=\int_0^\infty \frac{\mathrm{d}\mu_{m_n}(x,\hat{R}_1+\alpha\hat{R}_2)}{x-z}=m_n^{-1}\mathrm{Tr}\big[\hat{R}_1+\alpha\hat{R}_2-zI_{m_n}\big]^{-1},\ z=t+\mathrm{i}s,\ s>0$$

and the canonical equation for the matrix $C(z)=(c_{pl}(z))_{p,l=1}^{m_n}$

$$C(z,\alpha)=\left\{n_1^{-1}\sum_{k=1}^{n_1}\mathbf{E}\frac{\vec{\eta}_k\vec{\eta}_k^{\mathrm{T}}}{1+n_1^{-1}\vec{\eta}_k^{\mathrm{T}}C(z,\alpha)\vec{\eta}_k}+\alpha n_2^{-1}\sum_{k=1}^{n_2}\mathbf{E}\,\frac{\vec{\nu}_k\vec{\nu}_k^T}{1+n_2^{-1}\vec{\nu}_k^T C(z,\alpha)\vec{\nu}_k}-zI_m\right\}^{-1},$$

where $\vec{\eta}_k=\{\eta_{pk};\,p=1,\dots,m\}^{\mathrm{T}}=\vec{x}_k-\vec{a}_1$, $\vec{\nu}_k=\{\nu_{pk};\ p=1,\dots,m\}^T=\vec{y}_k-\vec{a}_2$ and I_m is the identity matrix, $s>0$. In [Gir84] it is shown that under some conditions with probability 1

$$\lim_{n_1,n_2\to\infty}\big[b(z,\alpha)-m^{-1}\mathrm{Tr}\,C(z,\alpha)\big]=0.$$

Using the proof of Theorem 3.1 we get that under some conditions

$$m^{-1}\mathrm{Tr}\,\hat{R}_1\left[\hat{R}_1 t+\hat{R}_2\right]^{-1}$$
$$=\int_0^\infty \frac{\mathrm{d}\mu_n\left(x,R_1,R_2\right)}{\alpha+\frac{t}{1+tmn_1^{-1}b_m(t,\alpha)}+x\left\{1+\left(\alpha-1\right)mn_2^{-1}+\frac{b_m(t,\alpha)}{1+tmn_1^{-1}b_m(t,\alpha)}\right\}},$$

where

$$b_m\left(t,\alpha\right)=m^{-1}\mathrm{Tr}\,\hat{R}_1\left[\hat{R}_1 t+\hat{R}_2\right]^{-1}.$$

We transform this expression as

$$b_m\left(t,\alpha\right)\left\{1+\left(\alpha-1\right)mn_2^{-1}+\frac{b_m\left(t,\alpha\right)}{1+tmn_1^{-1}b_m\left(t,\alpha\right)}\right\}$$
$$=\int_0^\infty \frac{\mathrm{d}\mu_n\left(x,R_1,R_2\right)}{\left\{1+\left(\alpha-1\right)mn_2^{-1}+\frac{b_m(t,\alpha)}{1+tmn_1^{-1}b_m(t,\alpha)}\right\}^{-1}\left[\alpha+\frac{t}{1+tmn_1^{-1}b_m(t,\alpha)}\right]+x}.$$

Now replace t by the function $\theta(t)$ which is the nonnegative solution of the equation

$$\left\{1+(\alpha-1)\,mn_2^{-1}+\frac{b_m(\theta(t),\alpha)}{1+tmn_1^{-1}b_m(\theta(t),\alpha)}\right\}^{-1}\left[\alpha+\frac{\theta(t)}{1+tmn_1^{-1}b_m(\theta(t),\alpha)}\right]=t.$$

Then we obtain

$$G_6=b_m(\theta(t),\alpha)\left\{1+(\alpha-1)\,mn_2^{-1}+\frac{b_m(\theta(t),\alpha)}{1+tmn_1^{-1}b_m(\theta(t),\alpha)}\right\}.$$

From [Gir44, p.218], [Gir54] we get under $t>0$

$$\operatorname*{p\lim}_{n_1,n_2\to\infty}\left[G_6-\int_0^\infty\frac{\mathrm{d}\mu_n(x,R_1,R_2)}{t+x}\right]=0,$$

or

$$\operatorname*{p\lim}_{n_1,n_2\to\infty}\left[G_6-m^{-1}\mathrm{Tr}R_1\left[R_1t+R_2\right]^{-1}\right]=0.$$

7. G_7-ESTIMATOR OF THE STATES OF DISCRETE CONTROL SYSTEMS

Now we briefly discuss some questions of GSA related to our topic. Increasing demands for the quality of operation of industrial robots led to the necessity of creating better methods of control that take into account dynamic characteristics of manipulators. In order to construct such control systems, it is necessary to have full knowledge of a mathematical model of the manipulator. The dynamic model of the manipulator is a system of nonlinear differential equations. Coefficients of these equations are connected in a rather complicated fashion via trigonometric functions with generalized coordinates of the manipulator. Such a system is complicated for practical use because of the essential nonlinearity and mutual influence of links. Therefore, a simplified mathematical model with adaptive adjustment of the parameters in the control process proves to be expedient.

7.1. Adaptive approach to the control of manipulator motion

The standard model was given by linear differential equations of the second order in which the desired characteristics of motion were pointed out. An adaptive regulator in accordance with the standard model "adjusts" control of the manipulator according to the desired motion.

Linearized with respect to the nominal motion, the mathematical model was used in a procedure of control synthesis on the basis of asymptotic linear regulators as well as for constructing autoregressive models, representing displacements in separate links. Parameters of the model are estimated in the process of motion, proceeding from the optimization of some quality criterion.

The dynamics are described by a Lagrange equation of the second kind, which depends on unknown parameters of the manipulator. Locally optimal finitely convergent methods of solving inequalities were used for adaptation algorithms. In [Gir54], a method of adaptive control of the manipulator without full knowledge of the mathematical model is proposed, and its characteristics are studied. The estimation of the parameters of the model is made by observations on the manipulator in the block of adaptation. Using these estimates, a linear regulator optimizing generalized energy is constructed. The estimate of the parameters and the controls is made recurrently. The algorithm proposed is locally optimal.

7.2. The discrete analog of the control system

The discrete analog of a mathematical model for the control of manipulator motion can be represented in the form

$$\vec{x}_{n+1} = A(\vec{x}_n)\vec{x}_n + B(\vec{x}_n)\vec{u}_n, \tag{7.1}$$

where $\vec{u}_n$ is the vector of control moments.

We define the trajectory of motion of the manipulator in the form of a sequence of points $\vec{a}_i \in R^{2m}, i = 1, 2, ...,$ through which the manipulator has to pass and approximate the dynamic model of the manipulator by a linear model

$$\vec{x}_{n+1} = A_n\vec{x}_n + B_n\vec{u}_n + \vec{\varepsilon}_{n+1}, \tag{7.2}$$

where A_n, B_n are unknown matrices, and $\vec{\varepsilon}_{n+1}$ are errors of modelling. Assume that the matrices $A(\vec{x}(t))$, $B(\vec{x}(t))$ in (3) are constant but unknown. Such assumption will be true for local displacements of the manipulator. Then (7.1) can be written in the form $\vec{x}_{n+1} = A\vec{x}_n + B\vec{u}_n$. We make $n > m$ observations of the manipulator under some fixed controls. From the observations, we construct estimators of the matrices $\hat{A}_n$, $\hat{B}_n$.

Using these estimators, we can find the extrapolated position of the manipulator

$$\vec{x}^e_{n+1} = \hat{A}_n\vec{x}_n + \hat{B}_n\vec{u}_n. \tag{7.3}$$

We choose the control $\vec{u}_n$ to minimize the functional

$$I_n\left(\tilde{\vec{u}}\right) = \inf_{\vec{u}_n}\left\{\left\|\vec{a}_{n+1} - \vec{x}^e_{n+1}\right\|^2 + \delta\left\|\vec{u}_n\right\|^2\right\},\ \delta > 0. \tag{7.4}$$

The observed position of the manipulator under this control will be

$$\vec{x}_{n+1} = \hat{A}_n\vec{x}_n + \hat{B}_n\tilde{\vec{u}}_n + \vec{\varepsilon}_{n+1}.$$

Without loss of generality, we assume that B is a known square matrix which has an inverse. The matrix A will be estimated by the least squares method

$$\hat{A}_n = \sum_{s=1}^{n}\left(\vec{x}_s - B\tilde{\vec{u}}_{s-1}\right)\vec{x}^T_{s-1}\left[\sum_{s=1}^{n}\vec{x}_{s-1}\vec{x}^T_{s-1}\right]^{-1}.$$

Controls from (7.4) will be given in the form (G_7–estimator)

$$\tilde{\vec{u}}_s = \left[\delta I + BB^T\right]^{-1}B^T\left(\vec{a}_{s+1} - \tilde{A}_s\vec{x}_s\right),$$

where

$$\tilde{A}_s = \tilde{A}_s\chi\left\{\left\|\tilde{A}_s\right\| < \|A\|\right\} + \tilde{A}_{s-1}\chi\left\{\left\|\tilde{A}_s\right\| \geq \|A\|\right\}$$

and $\chi\left\{\left\|\tilde{A}_s\right\| \geq \|A\|\right\}$ is the indicator of a random event. Given $\tilde{\vec{u}}_n$, we observe the vector $\vec{x}_{n+1}$ again, find $\tilde{\vec{u}}_{n+1}$, , and continue these calculations up to the moment of time s when $\left\|\vec{a}_s - \tilde{\vec{x}}_s\right\|^2 < \varepsilon$, where $\varepsilon > 0$ is a given number. We prove convergence of estimates of the matrix A.

7.3. The main assertion

THEOREM 7.1. [Gir54, p.518] *Let the following conditions hold:*

$$\mathbf{E}\left\{\vec{\varepsilon}_{n+1}/\vec{\varepsilon}_1, \dots, \vec{\varepsilon}_n\right\} = 0,\ n = 1, 2, \dots,$$

$$\sup_n \|\vec{a}_n\| < \infty,$$

$$\|A\| \left\{1 + \left\|\left[\delta I + BB^T\right]^{-1} BB^T\right\|\right\} < 1,$$

$$\sup_n \mathbf{E}\ \|\vec{\varepsilon}_n\|^4 < \infty,$$

$$\sup_n \left\|\left[n^{-1} \sum_{s=1}^n \mathbf{E}\,\vec{\varepsilon}_{s-1}\vec{\varepsilon}_{s-1}^T\right]^{-1}\right\| < \infty,$$

$$\limsup_{h\to\infty} \sup_{\substack{|i_p - j_p| \geq h, \\ p=1,\dots,3}} \left|\mathbf{P}\left\{\varepsilon_{i_p} < x_{i_p}, \varepsilon_{j_p} < x_{i_p}, p = 1, ..., 3\right\}\right.$$
$$\left. -\mathbf{P}\left\{\varepsilon_{i_p} < x_{i_p}, p = 1, \dots, 3\right\} \mathbf{P}\left\{\varepsilon_{j_p} < x_{i_p}, p = 1, \dots, 3\right\}\right| = 0.$$

Then

$$\lim_{n\to\infty} \mathbf{E}\left\|\vec{a}_s - \tilde{\vec{x}}_s\right\|^2 \leq c\delta \left\|\left[\delta I + BB^T\right]^{-1}\right\|^2,$$

and distribution functions of entries of matrix $\left[\hat{A}_n - A\right] n^{1/2}$ *are asymptotically normal.*

The proposed adaptive method was used for solving some control problem.

7.4. G-system of recursion equations

We will study estimators of parameters of systems with m_n unknown parameters and with the number n of observations satisfying the G-condition:

$$\limsup_{n\to\infty} m_n n^{-1} < \infty.$$

Namely

$$\vec{y}_k = \Theta \vec{y}_{k-1} + \vec{b}_{k-1} + \vec{\varepsilon}_k,$$

where $\Theta = \{\theta_{ij}\}_{i,j=1}^{m_n}$ is an unknown matrix, $\vec{y}_k$, $k = 1, 2, ...$ are m_n-dimensional observations, $\vec{y}_0$, $\vec{b}_{k-1}$, $k = 1, 2, ...$ are known vectors, $\vec{\varepsilon}_k$, $k = 1, 2, ...$ are m_n-dimensional random vectors. Note, that in the general case, the matrix $\sum_{k=1}^n \vec{y}_{k-1}\vec{y}_{k-1}^T$ can be degenerate. Therefore, we will search for an estimate of a matrix $\Theta = \{\theta_{ij}\}_{i,j=1}^{m_n}$ in regularized form:

$$\hat{\Theta}_n = \sum_{k=1}^n c_n^{-1}\left(\vec{y}_k - \vec{b}_{k-1}\right)\vec{y}_{k-1}^T \left[I_{m_n}\alpha + c_n^{-1} \sum_{k=1}^n \vec{y}_{k-1}\vec{y}_{k-1}^T\right]^{-1},$$

where $\alpha > 0$, and c_n is a certain sequence of numbers. Hence

$$\begin{aligned}\hat{\Theta}_n &= \Theta c_n^{-1} Y_n \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \\ &= \sum_{k=1}^{n} c_n^{-1} \vec{\varepsilon}_k \vec{y}_{k-1}^T \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1},\end{aligned}$$

where

$$Y_n = \sum_{k=1}^{n} \vec{y}_{k-1} \vec{y}_{k-1}^T.$$

Let us represent this estimator in the following form

$$\begin{aligned}&n^{-1}\mathrm{Tr}\, Q \left\{ \hat{\Theta}_n - \Theta c_n^{-1} Y_n \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\} \\ &= \mathrm{Tr}\, Q \sum_{k=1}^{n} c_n^{-1} \vec{\varepsilon}_k \vec{y}_{k-1}^T \left\{ n^{-1} \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} - \mathbf{E}\, n^{-1} \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\} \\ &\quad + \mathrm{Tr}\, Q \sum_{k=1}^{n} c_n^{-1} \vec{\varepsilon}_k \vec{y}_{k-1}^T \left\{ \mathbf{E}\, n^{-1} \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\},\end{aligned}$$

where $Q = \{q_{ij}\}_{i,j=1}^{m_n}$ is the matrix of real parameters.

7.5. Self-averaging of G-estimators

Let us find conditions of consistency of the G-estimator. We need some auxiliary statements.

LEMMA 7.1. [Gir44, p.220] *If the random vectors $\vec{\varepsilon}_k,\ k = 1, 2, \ldots$ are independent, $\|\Theta\| < 1,\ \mathbf{E}\,\vec{\varepsilon}_k = 0,\ k = 1, 2, \ldots,$ and*

$$\sup_n \max_{k=1,\ldots,n} \mathbf{E}\, \|\vec{\varepsilon}_k\|^2 c_n^{-1} < \infty,$$

then

$$\operatorname*{plim}_{n\to\infty} \left\{ n^{-1}\mathrm{Tr}\, \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} - \mathbf{E}\, n^{-1}\mathrm{Tr}\, \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\} = 0.$$

LEMMA 7.2. [Gir44, p.220] *If the random vectors $\vec{\varepsilon}_k,\ k = 1, 2, \ldots$ are independent, $\|\Theta\| < 1,\ \mathbf{E}\,\vec{\varepsilon}_k = 0,\ k = 1, 2, \ldots,$ and*

$$\sup_n \max_{k=1,\ldots,n} \mathbf{E}\, \|\vec{\varepsilon}_k\|^2 c_n^{-1} < \infty,$$

then

$$\operatorname*{plim}_{n\to\infty} \left\{ n^{-1}\mathrm{Tr}\, Q c_n^{-1} Y_n \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} - \mathbf{E}\, n^{-1}\mathrm{Tr}\, Q c_n^{-1} Y_n \left[I_{m_n}\alpha + c_n^{-1} Y_n \right]^{-1} \right\} = 0.$$

Thus, if the conditions of Lemma 7.2 are satisfied and random variables

$$\left\|\sum_{k=1}^{n} c_n^{-1}\vec{\varepsilon}_k \vec{y}_{k-1}^T\right\|$$

are bounded in probability, then

$$\frac{1}{n}\operatorname{Tr} Q\left\{\hat{\Theta}_n - \Theta \mathbf{E}\,\frac{1}{c_n}Y_n\left[I_{m_n}\alpha + \frac{1}{c_n}Y_n\right]^{-1}\right\}$$
$$\cong \mathbf{E}\,\frac{1}{n}\operatorname{Tr} Q\sum_{k=1}^{n}\frac{1}{c_n}\vec{\varepsilon}_k\vec{y}_{k-1}^T\left[I_{m_n}\alpha + \frac{1}{c_n}Y_n\right]^{-1}.$$

Suppose, the matrix $R = \mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}$ is nondegenerate. Then we consider the G_7-estimator

$$\hat{\Theta}_n\left\{\mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}\right\}^{-1}$$

of matrix Θ.

Theorem 7.2. [Gir44, p.220] *If the random vectors $\vec{\varepsilon}_k,\ k = 1, 2, \ldots$ are independent, $\|\Theta\| < 1,\ \mathbf{E}\,\vec{\varepsilon}_k = 0,\ k = 1, 2, \ldots,$ and*

$$\sup_n \max_{k=1,\ldots,n} \mathbf{E}\,\|\vec{\varepsilon}_k\|^2 c_n^{-1} < \infty,$$

$$\limsup_{n\to\infty}\left\|\sum_{k=1}^{n} c_n^{-1}\vec{\varepsilon}_k\vec{y}_{k-1}^T\right\| < \infty,$$

$$\limsup_{n\to\infty}\left\|\left\{\mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}\right\}^{-1}\right\| < \infty$$

then

$$p\lim_{n\to\infty}\left\|\hat{\Theta}_n\left\{\mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}\right\}^{-1} - \Theta\right\| = 0.$$

Suppose, the matrix

$$R = \mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}$$

is nondegenerate. We consider the G_7-estimator

$$\hat{\Theta}_n\left\{\mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}\right\}^{-1}$$

of the matrix Θ.

Theorem 7.3. [Gir44, p.220] *If the random vectors $\vec{\varepsilon}_k,\ k = 1, 2, \ldots$ are independent, $\|\Theta\| < 1,\ \mathbf{E}\,\vec{\varepsilon}_k = 0,\ k = 1, 2, \ldots,$ and*

$$\sup_n \max_{k=1,\ldots,n} \mathbf{E}\,\|\vec{\varepsilon}_k\|^2 c_n^{-1} < \infty,$$

$$p\lim_{n\to\infty}\sup\left\|\sum_{k=1}^{n} c_n^{-1}\vec{\varepsilon}_k\vec{y}_{k-1}^T\right\| < \infty,$$

$$\lim_{n\to\infty}\sup\left\|\left\{\mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}\right\}^{-1}\right\| < \infty,$$

then

$$p\lim_{n\to\infty}\left\|\hat{\Theta}_n\left\{\mathbf{E}\,c_n^{-1}Y_n\left[I_{m_n}\alpha + c_n^{-1}Y_n\right]^{-1}\right\}^{-1} - \Theta\right\| = 0.$$

8. CLASS OF G_8-ESTIMATORS OF THE SOLUTIONS OF SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS (SLAE)

Let some system S with input vector $\vec{x}^T = (x_1, \ldots, x_m)$ and output variables y be given. As a mathematical model M_1 of the system S, it is natural to take the equation $y = A(\vec{x}) + \varepsilon$ where $A(\vec{x})$ is some operator, and ε is an error of such representation. Choosing different input vectors $\vec{x}_1, \ldots, \vec{x}_n$ we have a system of equations

$$\vec{y} = A + \vec{\varepsilon},$$

where

$$A = \{A(\vec{x}_1), \ldots, A(\vec{x}_n)\}$$

is an operator acting in a space of vectors $\vec{x}$ with values in a space of vectors $\vec{y}^T = (y_1, \ldots, y_n)$, and $\vec{\varepsilon}^T = \{\varepsilon_1, \ldots, \varepsilon_n\}$ is a vector of errors of the model M_1. If $y = f(\vec{x})$, where f is an unknown analytic function, then for simplification of the calculations we can take the operator A to be

$$A\vec{x} = \sum_{i=1}^{m} c_i x_i; \quad A\vec{x} = \sum_{i=1}^{m} c_i \varphi_i(x_i)$$

or

$$A\vec{x} = \sum_{i=1}^{m} c_i x_i + \sum_{i,j=1}^{m} c_{ij} x_i x_j + \ldots + \sum_{i_1,\ldots,i_k=1}^{m} c_{i_1,\ldots,i_k} x_{i_1} \ldots x_{i_k},$$

where $c_i, \; c_{ij}, \; c_{i_1,\ldots,i_k}$ are unknown coefficients; φ_i are known functions. We note that in all these cases $A\vec{x} = \vec{c}^T\vec{z}$, where $\vec{c}$ is an unknown vector and $\vec{z}$ is a known vector. Thus, we arrive at model M_1 which is linear with respect to the unknown parameters:

$$\vec{y} = X\vec{c} + \tilde{\vec{\varepsilon}},$$

where $X^T = [\vec{z}_1, \ldots, \vec{z}_n]$ and $\tilde{\vec{\varepsilon}}$ is a vector of errors. In this section we formulate the methods of finding coefficients c_i if we have the observations of y and the input vectors $\vec{x}$.

8.1. The classical least squares method

Assume that a mathematical model of a system S has the form

$$y = \vec{x}^T \vec{c} + \varepsilon,$$

where $\vec{x}$ is an m-dimensional vector of input parameters, $\vec{c}$ is an unknown m-dimensional vector; y is the observable variable of a system S, and ε is a model error. Let n observations $y_1, \ldots, y_n$ of a system S under the values $\vec{x}_1, \ldots, \vec{x}_n$ of a vector $\vec{x}$ be made. Then for the unknown vector $\vec{c}$ we get the system of equations

$$\vec{y} = X\vec{c} + \vec{\varepsilon}, \tag{8.1}$$

where $\vec{\varepsilon}^T = (\varepsilon_1, \ldots, \varepsilon_n)$ is the observation error. The vectors $\vec{c}$ and $\vec{\varepsilon}$ in the system of equations (8.1) are unknown. This system of equations is undetermined with respect to the unknown vectors $\vec{c}$ and ε and in the general case has an infinite set of solutions. Calculating the vector $\vec{c}$ it is desirable to know the value of the vector $\vec{\varepsilon}$. However, because of the indeterminancy of the system (8.1), it is difficult to find the true value of the vector $\vec{\varepsilon}$ without any auxiliary conditions. We can reduce the system (8.1) to the form

$$\vec{y} = X\hat{\vec{c}}, \tag{8.2}$$

where the vector $\vec{c}$ is replaced by a new vector $\hat{\vec{c}}$ which is different from $\vec{c}$ in general. The preliminary investigations of the system (8.1) were made in the following way. In general the solution $\hat{\vec{c}}$ of a system (8.2) may not exist. However it is not necessary to find a solution of this system. We need to find the value of $\hat{\vec{c}}$ which minimizes some quality criterion of an estimator $F\left\{\vec{y} - X\hat{\vec{c}}\right\}$. For the simplification of calculations as the quality criterion the function

$$I(\vec{u}) = \vec{u}^T \vec{u} = \|\vec{u}\|^2$$

is usually chosen. If the inverse matrix $\left(X^T X\right)^{-1}$ exists, then we can obtain the minimizer of $\left\|\vec{y} - X\hat{\vec{c}}\right\|$ as

$$\hat{\vec{c}} = (X^T X)^{-1} X^T \vec{y}. \tag{8.3}$$

This formula explains the name, the "Least Squares Method". The estimation is

$$\hat{\vec{c}} - \vec{c} = (X^T X)^{-1} X^T \vec{\varepsilon}. \tag{8.4}$$

If the inverse matrix $\left(X^T X\right)^{-1}$ does not exist, then the function $\varphi(\vec{c}) := \|\vec{y} - X\vec{c}\|^2$ can have uncountable points of minimum. Again, to simplify calculations among all points of the minimum, the vector $\tilde{\vec{c}}$ with the smallest Euclidean norm is chosen. We can find this vector in the following way: consider the function

$$\varphi(\vec{c}, \alpha) := \|\vec{y} - X\vec{c}\|^2 + \alpha \|\vec{c}\|^2, \ \ \alpha > 0$$

instead of the function $\varphi(\vec{c}) := \|\vec{y} - X\vec{c}\|^2$. Because $\alpha > 0$, the minimum of function $\varphi(\vec{c}, \alpha)$ is unique and the vector $\vec{c}_\alpha$, under which the function $\varphi(\vec{c}, \alpha)$ will take the minimal value, is defined by the formula

$$\vec{c}_\alpha = \left(\alpha I + X^T X\right)^{-1} X^T \vec{y}. \tag{8.5}$$

It is easy to prove that $\lim_{\alpha \downarrow 0} \vec{c}_\alpha = \tilde{\vec{c}}$.

As the G-estimators of the regularized pseudo-solutions

$$\vec{x}_\alpha = \left[I\alpha + X^T X\right]^{-1} X^T \vec{b},$$

we choose a regularized solution

$$\vec{y}_\theta = \mathrm{Re}\left[I\left(\theta + \mathrm{i}\varepsilon\right) + \Xi^T \Xi\right]^{-1} \Xi^T \vec{b},$$

where $\varepsilon \neq 0$ and θ are real parameters, $\Xi = \left(\xi_{ij}^{(n)}\right)$ is the observation of the random matrix $X + H$, where H is a certain random matrix. The G-estimators of the values $\vec{x}_\alpha$ belong to the class of $\tilde{G}_8$ -estimators and are denoted by G_8. In this section, the following G_8-estimator of $\tilde{G}_8$-class is proposed

$$\vec{G}_8 = \mathrm{Re}\left[I\left(\hat{\theta}_1 + \mathrm{i}\varepsilon\right) + \Xi^T \Xi\right]^{-1} \Xi^T \vec{b}. \tag{8.6}$$

Here $\hat{\theta}_1$ is the maximal real solution of the equation

$$f_n(\theta) = \alpha, \tag{8.7}$$

where $\alpha \geq 0$,

$$f_n(\theta) = \theta \mathrm{Re}\left[1 + \delta_1 a\left(\theta\right)\right]^2 - \varepsilon \mathrm{Im}\,\left[1 + \delta_1 a\left(\theta\right)\right]^2 + \left(\delta_1 - \delta_2\right)\left[1 + \delta_1 \mathrm{Re} a(\theta)\right],$$

$$a\left(\theta\right) = \frac{1}{n}\mathrm{Tr}\left[I\left(\theta + \mathrm{i}\varepsilon\right) + \Xi^T \Xi\right]^{-1},\ \delta_1 = \sigma_n^2 n\ ,\ \delta_2 = \sigma_n^2 m,$$

σ_n^2 is the variance of entries $\xi_{ij}^{(n)}$ of the matrix $\Xi = \left(\xi_{ij}^{(n)}\right)$. We call equation (8.7) the main equation for the G_8-estimator.

It is proved [Gir44, Gir54, Gir69, Gir84] that under certain conditions, for every $\gamma > 0$

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbf{P}\left\{|\vec{d}[\vec{G}_8 - (I\alpha + X^T X)^{-1} X^T \vec{b}]| > \gamma\right\} = 0,$$

where $\vec{d}$ is an arbitrary vector such that $\vec{d}^T \vec{d} \leq c < \infty$.

8.2. Modified G_8-estimator of the solution of SLAE

In this section, the following modified G_8-estimator from the $\tilde{G}_8$-class for

$$\vec{x}_\alpha = [I\alpha + A^T A]^{-1} A^T \vec{b}$$

is proposed,

$$\vec{G}_8(\alpha, \varepsilon, B, C) = \mathrm{Im} \int_0^B \left\{ \frac{\mathrm{e}^{|sp|}}{\pi} \int_{-C}^{C} \mathrm{Im}\,([I\hat{\theta} + X^T X]^{-1} X^T \vec{b}) \mathrm{e}^{-\mathrm{i}tp} \mathrm{d}t \right\} \mathrm{e}^{-p(\alpha - \mathrm{i}\varepsilon)} \mathrm{d}p.$$

Here $\hat{\theta}_1$ is the measurable complex solution of the equation

$$\hat{\theta}\left\{1 + \frac{\sigma^2}{n}\mathrm{Tr}\left[I\hat{\theta} + X^T X\right]^{-1}\right\}^2 + \left(1 - \frac{m_n}{n}\right)\left\{1 + \frac{\sigma^2}{n}\mathrm{Tr}\left[I\hat{\theta} + X^T X\right]^{-1}\right\} = -z,$$

σ_n^2 is the variance of entries $x_{ij}^{(n)}$ of observation $X = \left(x_{ij}^{(n)}\right)$ of matrix $A + \Xi$, $z = t + \mathrm{i}s$, $s \geq c$, c is a certain constant.

Under certain conditions we have ([Gir44], [Gir54], [Gir69], [Gir84])

$$\lim_{B\to\infty} \lim_{C\to\infty} p\lim_{n\to\infty} \vec{d}^T \left[\vec{G}_8(\alpha, \varepsilon, B, C) - \mathrm{Re}\left[I(\alpha + \mathrm{i}\varepsilon) + A^T A\right]^{-1} A^T \vec{b}\right] = 0.$$

8.3. G_8-estimator of the solutions of SLAE with block structure

For linear forms $\vec{d}^T \vec{x}_\alpha$ of regularized pseudo-solutions $\vec{x}_\alpha = [I\alpha + A^T A]^{-1} A^T \vec{b}$ of the systems of linear algebraic equations $A\vec{x} = \vec{b}$ with block structure, the following G_8-estimator

$$\vec{d}^T \vec{G}_8 = -\mathrm{Re}\vec{d}^T \left[C_1 + \mathrm{i}\varepsilon I_m + Z_s^T \left(C_2 - \mathrm{i}\varepsilon I_n\right)^{-1} Z_s\right]^{-1} Z_s^T \left(C_2 - \mathrm{i}\varepsilon I_n\right)^{-1} \vec{b},$$

is suggested. Here A is a matrix of the size $np \times mq$, $n \geq m$, $\vec{x}$ and $\vec{b}$ are vectors, $\alpha > 0$ is a parameter of regularization, $\varepsilon > 0$; $\vec{b} \in R^{np}$; $\vec{d}^T \in R^{mq}$; $Z_s = s^{-1} \sum_{i=1}^s X_i$; X_i are independent observations of the matrix $A + \Xi$, $\Xi = \left(\Xi_{ij}^{(n)}\right)_{i=1,\dots,n}^{j=1,\dots,m}$ is a random matrix with independent blocks $\Xi_{ij}^{(n)}$, $\mathbf{E}\,\Xi_{ij}^{(n)} = 0$, $\mathbf{E}\left\|\Xi_{ij}^{(n)}\right\|^2 < \infty$; and $C_1 = \left(C_{1i}\delta_{ij}\right)_{i,j=1}^m$, $C_2 = \left(C_{2i}\delta_{ij}\right)_{i,j=1}^n$ are block diagonal real matrices that are arbitrary measurable solutions of the system of nonlinear equations

$$C_{1l} + \mathrm{Re}\sum_{j=1}^n \left[\frac{1}{s}\mathbf{E}\,\Xi_{jl}^{(n)T}\left\{Q_{jj}\right\}\Xi_{jl}\right]_{Q=\left[C_2 - \mathrm{i}\varepsilon I_n + \tilde{X}(C_1 + \mathrm{i}\varepsilon I_m)^{-1}\tilde{X}^T\right]^{-1}} = \alpha I;$$

$$C_{2k} + \mathrm{Re}\sum_{j=1}^m \frac{1}{s}\left[\mathbf{E}\,\Xi_{kj}^{(n)}\left\{\Theta_{jj}\right\}\Xi_{kj}^T\right]_{\Theta=\left[C_1 + \mathrm{i}\varepsilon I_m + \tilde{X}^T(C_2 - \mathrm{i}\varepsilon I_n)^{-1}\tilde{X}\right]^{-1}} = I,$$

$$k = 1, \dots, n;\ p = 1, \dots, m,\ \tilde{X} = Z_s.$$

It is proved [Gir84, p.236] that under certain conditions, for every $\gamma > 0$,

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \mathbf{P}\left\{\left|\vec{d}^T\left(\vec{G}_8 - \vec{x}_\alpha\right)\right| > \gamma\right\} = 0.$$

8.4. G_8-estimator of the solutions of SLAE with symmetric block structure

Let $A\vec{x} = \vec{b}$ be a SLAE, where $A_{pq\times pq} = \left(A_{ks}^{(n)}\right)_{k,s=1}^p$, $A_{ks}^{(n)} = A_{ks}^{(n)T}$ and $A_{ks}^{(n)}$; $k \geq s$, $k, s = 1, \dots, p$ are blocks of the dimension q, and let $\vec{x}$, $\vec{b}$ be vectors. We consider the linear form of the regularized solution of such a system

$$\vec{d}^T \vec{x}_\varepsilon = \vec{d}^T \mathrm{Re}\left[A_{pq\times pq} + \mathrm{i}\varepsilon I_n\right]^{-1} \vec{b};\ \ \vec{d} \in R^n;\ n = pq;\ \varepsilon > 0.$$

For linear forms $\vec{d}^T \vec{x}_\varepsilon$ of regularized pseudo-solutions,

$$\vec{x}_\varepsilon = \mathrm{Re}\left[A_{pq\times pq} + \mathrm{i}\varepsilon I_n\right]^{-1} \vec{b},$$

of the systems of linear algebraic equations $A\vec{x} = \vec{b}$ with block structure, the following G_8–estimator

$$\vec{d}^T \vec{G}_8 = -\mathrm{Re}\left[X_{pq\times pq} + C(\varepsilon) + \mathrm{i}\varepsilon I_n\right]^{-1} \vec{b}$$

is considered. Here, $X_{pq\times pq}$ is an observation of matrix $\Xi_{pq\times pq} + A_{pq\times pq}$, $\Xi_{pq\times pq} = \left(\Xi_{ks}^{(n)}\right)_{k,s=1}^{p}$, $\Xi_{ks}^{(n)} = \Xi_{ks}^{(n)T}$ and $\Xi_{ks}^{(n)}$; $k \geq s$, $k.s = 1, ..., p$ are independent random blocks of the dimension q, $C_{pq\times pq}(\varepsilon) = \left(\delta_{ij} C_{jj}^{(n)}(\varepsilon)\right)_{i,j=1}^{p}$ and the matrix-blocks $C_{ss}(\varepsilon)$ satisfy for $z = \mathrm{i}\varepsilon$ the canonical equation

$$C_{jj}(\varepsilon) = \mathrm{Re}\,\mathbf{E} \sum_{s=1}^{p} \Xi_{js}^{(n)} Q_{ss}\, \Xi_{js}^{(n)T} \Big|_{Q=\left[X_{pq\times pq} + C_{pq\times pq}(\varepsilon) + \mathrm{i}\varepsilon I_n\right]^{-1}}.$$

It is proven in [Gir84, p.250] that under certain conditions, for every $\gamma > 0$

$$\lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \mathbf{P}\left\{\left|\vec{d}^T\left(\vec{G}_8 - \vec{x}_\varepsilon\right)\right| > \gamma\right\} = 0.$$

9. G_9-ESTIMATOR OF THE SOLUTION OF THE DISCRETE KOLMOGOROV-WIENER FILTER

The discrete analog of the Kolmogorov-Wiener filter has the form

$$R_m \vec{\varphi} = \vec{b}, \tag{9.1}$$

where

$$R_m = \left\{m^{-1} R\left(sm^{-1}, km^{-1}\right)\right\}_{k,s=1}^{m};\ \ \vec{b}^T(t) = \left\{Q\left(t, sm^{-1}\right),\ \ s = 1, \dots, m\right\},$$

$$\vec{\varphi}^T(t) = \left\{\varphi\left(t, km^{-1}\right),\ \ k = 1, ..., m\right\},$$

$$R(x, y) = \mathbf{E}\left[\alpha(x) - \mathbf{E}\,\alpha(x)\right]\left[\alpha(y) - \mathbf{E}\,\alpha(y)\right],$$

$$Q(x, y) = \mathbf{E}\left[\alpha(x) - \mathbf{E}\,\alpha(x)\right]\left[\beta(y) - \mathbf{E}\,\beta(y)\right],$$

and $\alpha(x)$, $\beta(y)$ are random processes. If $R_m > 0$, then the estimator $\hat{\vec{\varphi}} = \left(\hat{R}_m\right)^{-1} \hat{\vec{b}}$ converges in probability to $\vec{\varphi}$ when $n_1,\ n_2 \to \infty$, where

$$\hat{R} = \left\{m^{-1} \hat{R}\left(sm^{-1}, km^{-1}\right)\right\}_{k,s=1}^{m},\ \ \vec{\varphi}^T(t) = \left\{\varphi\left(t, km^{-1}\right),\ \ k = 1, ..., m\right\};$$

$$\hat{\vec{b}}^T(t) = \left\{\hat{Q}\left(t, sm^{-1}\right),\ \ s = 1, \dots, m\right\},$$

$$\hat{R}(x, y) = (n_1 - 1)^{-1} \sum_{k=1}^{n_1} \left[\alpha_k(x) - \hat{\alpha}(x)\right]\left[\alpha_k(y) - \hat{\alpha}(y)\right],$$

$$\hat{Q}(x,y) = (n_2-1)^{-1} \sum_{k=1}^{n_2} \left[\alpha_k(x) - \hat{\alpha}(x)\right] \left[\beta_k(y) - \hat{\beta}(y)\right],$$

and $\alpha_k(x)$, $\beta_k(y)$ are independent observations of $\alpha(x)$, $\beta(y)$.

As mentioned in previous sections of this chapter, the large order of system (9.1) requires a large number of observations of stochastic processes $\alpha(x)$, $\beta(y)$.

Therefore, it is of interest to obtain more accurate estimators. Applying the G-analysis technique, which is described in [Gir44, Gir54, Gir69, Gir84], we can obtain an estimator of $\vec{\varphi}$, such that it would approach in probability $\vec{\varphi}$, provided that $\lim_{n\to\infty} mn^{-1} = c < 1$. We assume for simplification of formulas that vector $\vec{b}$ is known. This estimator will be referred to as the G_9-estimator. It is

$$\vec{G}_9 = \left(\hat{R}_m\right)^{-1} \vec{b}\left(1 - \frac{m_n}{n}\right). \tag{9.2}$$

Denote

$$\vec{\alpha}_k^{\mathrm{T}} = \left(\alpha_k\left(\frac{s}{m}\right),\ s=1,\dots,m\right), \qquad R^{-1/2}(\vec{\alpha}_k - \mathbf{E}\,\vec{\alpha}_k) = \vec{\xi}_k = (\xi_{sk}\ s=1,\dots,m)^T.$$

THEOREM 9.1. ([Gir44], [Gir54], [Gir69], [Gir84]) *If random variables ξ_{sk} are independent for every n, $\mathbf{E}\,|\xi_{ki}|^{4+\delta} \le c,\ \delta > 0,\ \lim\limits_{n_1\to\infty} mn_1^{-1} < 1$; $\lambda_i(R_m) \le c < \infty$, the vector $\vec{b}$ is known,*

$$\sup_m \left[\vec{b}^T\vec{b} + \vec{c}^T\vec{c}\right] < \infty,$$

where $\vec{c} \in R^m$, and $\lambda_i(R_m)$ are the eigenvalues of the matrix R_m, then

$$\operatorname*{p\,lim}_{n_1\to\infty} \left[\vec{c}^T\vec{G}_9 - \vec{c}^T\vec{\varphi}\right] = 0.$$

10. G_{10}-ESTIMATOR OF THE SOLUTION OF A REGULARIZED DISCRETE KOLMOGOROV-WIENER FILTER WITH KNOWN FREE VECTOR

The discrete analog of a regularized Kolmogorov-Wiener filter has the form

$$(\varepsilon I_m + R_m)\,\vec{\varphi}(t) = \vec{b}(t),$$

where $\varepsilon > 0$ is a parameter of regularization,

$$R_m = \left\{m^{-1}R\left(sm^{-1}, km^{-1}\right)\right\}_{k,s=1}^m;\ \ \vec{b}^T(t) = \left\{Q\left(t, sm^{-1}\right),\ \ s=1,\dots,m\right\},$$

$$\vec{\varphi}^T(t) = \left\{\varphi\left(t, km^{-1}\right),\ \ k=1,\dots,m\right\},$$

$$R(x,y) = \mathbf{E}\left[\alpha(x) - \mathbf{E}\,\alpha(x)\right]\left[\alpha(y) - \mathbf{E}\,\alpha(y)\right],$$

$$Q(x,y) = \mathbf{E}\left[\alpha(x) - \mathbf{E}\,\alpha(x)\right]\left[\beta(y) - \mathbf{E}\,\beta(y)\right].$$

Here $\alpha(x)$, $\beta(y)$ are random processes. The estimator $\hat{\vec{\varphi}(t)} = \left(\varepsilon I + \hat{R}_m\right)^{-1} \hat{\vec{b}(t)}$ converges in probability to $\vec{\varphi}(t)$ when $n_1,\ n_2 \to \infty$. Here

$$\hat{R} = \left\{m^{-1}\hat{R}\left(sm^{-1}, km^{-1}\right)\right\}_{k,s=1}^{m},\ \ \vec{\varphi}^T(t) = \left\{\varphi\left(t, km^{-1}\right),\ \ k = 1, ..., m\right\};$$

$$\hat{\vec{b}}^T(t) = \left\{\hat{Q}\left(t, sm^{-1}\right),\ \ s = 1, \dots, m\right\},$$

$$\hat{R}(x,y) = (n_1 - 1)^{-1}\sum_{k=1}^{n_1}\left[\alpha_k(x) - \hat{\alpha}(x)\right]\left[\alpha_k(y) - \hat{\alpha}(y)\right],$$

$$\hat{Q}(x,y) = (n_2 - 1)^{-1}\sum_{k=1}^{n_2}\left[\alpha_k(x) - \hat{\alpha}(x)\right]\left[\beta_k(y) - \hat{\beta}(y)\right],$$

and $\alpha_k(x)$, $\beta_k(y)$ are independent observations of $\alpha(x)$, $\beta(y)$. Applying the G-analysis technique, which is described in [Gir44, Gir54, Gir69, Gir84], we can obtain an estimator of $\vec{\varphi}(t)$, which approaches in probability $\vec{\varphi}(t)$, provided that

$$\lim_{n_1\to\infty} mn_1^{-1} < 1; \quad \lim_{n_1\to\infty} mn_2^{-1} < \infty$$

This estimator will be referred to as the G_{10}-estimator:

$$\vec{G}_{10} = \varepsilon^{-1}\left(I + \hat{\theta}\hat{R}_m\right)^{-1}\vec{b}(t), \tag{10.1}$$

where $\hat{\theta}$ is a nonnegative solution of the equation

$$\theta\left[1 - \gamma_{n_1} + \gamma_{n_1} m^{-1}\mathrm{Tr}\ \left(\theta I + \hat{R}_m\right)^{-1}\right] = \varepsilon^{-1},\ \ \varepsilon > 0;\ \ \gamma_{n_1} = mn_1^{-1} < 1. \tag{10.2}$$

THEOREM 10.1. [Gir44, Gir54, Gir69, Gir84] *Assume that*

$$\vec{x}_k := \left\{\alpha_k\left(sm^{-1}\right);\ \ s = 1, ..., m\ \right\}^T = R_m^{1/2}\vec{\eta}_k + \vec{a},$$

$$\left\{\vec{\eta}_k^T = \{\eta_{ik};\ i = 1, ..., m\}\right\};\ k = 1, ..., n$$

$R_m^{1/2}$ *is a symmetric matrix,* t *is fixed,* $n_1 = n_2 = n$*, random variables* $\eta_{ik};\ i = 1, ..., m;\ k = 1, ..., n$ *are independent for every* n*, and*

$$\mathbf{E}\,\eta_{ik} = 0; \quad \mathbf{E}\,\eta_{ik}^2 = 1;\ \ i = 1, \dots, m; \quad k = 1, \dots, n$$

$\lim_{n\to\infty} mn^{-1} < 1,\ \lambda_i(R) \le c < \infty$, *the vector* $\vec{b}$ *is known,*

$$\sup_m\left[\vec{b}^T\vec{b} + \vec{c}^T\vec{c}\right] < \infty,\ \varepsilon > 0,$$

where $\vec{c} \in R^m,\ \lambda_i(R)$ *are the eigenvalues of the matrix* R_m*. Then*

$$\underset{n\to\infty}{p\lim}\left[\vec{c}^T G_{10} - \vec{c}^T \vec{\varphi}\right] = 0.$$

10.1. G_{10}-estimator for the solution of a Kolmogorov-Wiener filter with unknown vector

Consider the discrete analog of a regularized Kolmogorov-Wiener filter

$$\vec{b}(t) = (\varepsilon I + R_m)\,\vec{\varphi}(t), \tag{10.3}$$

where $\varepsilon > 0$ is a parameter of regularization,

$$R_m = \left\{m^{-1} R\left(sm^{-1}, km^{-1}\right)\right\}_{k,s=1}^{m};\ \ \vec{b}^T(t) = \left\{Q\left(t, sm^{-1}\right),\ \ s = 1, ..., m\right\},$$

$$\vec{\varphi}^T(t) = \left\{\varphi\left(t, km^{-1}\right),\ \ k = 1, ..., m\right\},$$

$$R(x, y) = \mathbf{E}\left[\alpha(x) - \mathbf{E}\,\alpha(x)\right]\left[\alpha(y) - \mathbf{E}\,\alpha(y)\right],$$

$$Q(x, y) = \mathbf{E}\left[\alpha(x) - \mathbf{E}\,\alpha(x)\right]\left[\beta(y) - \mathbf{E}\,\beta(y)\right].$$

For this case, when free vector $\vec{b}(t)$ is unknown, the estimator vector $\vec{\varphi}^T(t) = \{\varphi\left(t, km^{-1}\right), k = 1, \ldots, m\}$ will be referred to as the $\tilde{\vec{G}}_{10}$ -estimator. It has the form

$$\tilde{\vec{G}}_{10} = \varepsilon^{-1}\left\{1 + \varepsilon\hat{\theta}\left[\gamma_n - n^{-1}\mathrm{Tr}\left\{I + \hat{\theta}\hat{R}_m\right\}^{-1}\right]\right\}\left(I + \hat{\theta}\hat{R}_m\right)^{-1}\hat{\vec{b}}, \tag{10.4}$$

where $\hat{\theta}$ is a nonnegative solution of the equation

$$\theta\left[1 - \gamma_n + \gamma_n m^{-1}\mathrm{Tr}\left(\theta I + \hat{R}_m\right)^{-1}\right] = \varepsilon^{-1},\ \ \varepsilon > 0;\ \ \gamma_n = mn^{-1} < 1, \tag{10.5}$$

$$\hat{R}_m = n^{-1}\sum_{k=1}^{n} R_m^{1/2}\vec{\eta}_k\vec{\eta}_k R_m^{1/2} - \left(\hat{\vec{x}} - \vec{a}\right)\left(\hat{\vec{x}} - \vec{a}\right)^T;\ \ \hat{\vec{x}} = n^{-1}\sum_{k=1}^{n}\vec{x}_k,$$

$$\hat{\vec{b}} = n^{-1}\sum_{k=1}^{n}(y_k - \hat{y})\left(\vec{x}_k - \hat{\vec{x}}\right);\ \ \hat{y} = n^{-1}\sum_{k=1}^{n} y_k,$$

$$\vec{x}_k := \left\{\alpha_k\left(sm^{-1}\right);\ \ s = 1, ..., m\right\}^T = R_m^{1/2}\vec{\eta}_k + \vec{a};\ \ y_k := \beta_k(t) = \xi_k + p,$$

$R_m^{1/2}$ is a symmetric matrix, t is fixed, $n_1 = n_2 = n$, the vectors

$$\left\{\vec{\eta}_k^T = \{\eta_{ik};\ i = 1, ..., m\};\ \ \xi_k\right\};\ \ k = 1, ..., n$$

are independent for every n, random variables $\eta_{ik};\ i = 1, ..., m$ are independent; $\xi_k;\ k = 1, ..., n$ are also independent, and

$$\mathbf{E}\,\eta_{ik}=0;\ \mathbf{E}\,\eta_{ik}^2=1;\ \ \mathbf{E}\ \xi_k=0;\ \mathbf{E}\,\xi_k\left(\sqrt{R_m}\vec{\eta}_k\right)_{ik}=b_i;\ i=1,...,m;\quad k=1,...,n.$$

THEOREM 10.3. [Gir84, p.298] *If*

$$\limsup_{n\to\infty} mn^{-1}<1,$$

$$\lambda_i\left(R_m\right)\le c<\infty;\ i=1,...,m$$

$$\sup_m\left[\vec{b}^T\vec{b}+\vec{c}^T\vec{c}\right]<\infty,\ \varepsilon>0$$

$$\sup_n \max_{i=1,\dots,m;\ k=1,\dots,n}\mathbf{E}\,\left|\eta_{ik}\right|^4<\infty,$$

$$\sup_n \max_{k=1,\dots,n}\max_{s=1,\dots,m}\lambda_s\left\{\mathbf{E}\,\left[\xi_k\sqrt{R_m}\vec{\eta}_k-\vec{b}\right]\left[\xi_k\sqrt{R_m}\vec{\eta}_k-\vec{b}\right]^T\right\}<\infty,$$

then for every $\varepsilon>0$

$$\underset{n\to\infty}{p\lim}\left[\vec{c}^T\tilde{\vec{G}}_{10}-\vec{c}^T\vec{\varphi}\right]=\underset{n\to\infty}{p\lim}\left[\vec{c}^T\tilde{\vec{G}}_{10}-\vec{c}^T\left(I\varepsilon+R_m\right)^{-1}\vec{b}\right]=0.$$

11. G_{11}-ESTIMATOR OF THE MAHALANOBIS DISTANCE

Let $\vec{x}_i,\ \vec{y}_j;\ i=1,...,n_1;\ j=1,...,n_2$ be independent observations of m-dimensional random vectors $\vec{\xi}=\vec{a}_1+\sqrt{R}\vec{\mu},\ \vec{\mu}^T=\{\mu_i,\ i=1,\dots,m\}$ and $\vec{\eta}=\vec{a}_2+\sqrt{R}\vec{\nu},\ \vec{\nu}^T=\{\nu_i,\ i=1,\dots,m\}$ respectively and suppose that random variables $\mu_i,\ \nu_i,\ i=1,\dots,m$ are independent for every n. As the empirical mean value vectors and the covariance matrix R, we take:

$$\hat{\vec{a}}_1=n_1^{-1}\sum_{i=1}^{n_1}\vec{x}_i,\ \hat{\vec{a}}_2=n_2^{-1}\sum_{i=1}^{n_2}\vec{y}_i,$$

$$\hat{R}=\frac{1}{n_1+n_2-2}\left\{\sum_{i=1}^{n_1}\left(\vec{x}_i-\hat{\vec{a}}_1\right)\left(\vec{x}_i-\hat{\vec{a}}_1\right)^T+\sum_{i=1}^{n_2}\left(\vec{y}_i-\hat{\vec{a}}_2\right)\left(\vec{y}_i-\hat{\vec{a}}_2\right)^T\right\}.$$

We shall refer to the expression

$$G_{11}=\left\{\left(\hat{\vec{a}}_1-\hat{\vec{a}}_2\right)^T\hat{R}^{-1}\left(\hat{\vec{a}}_1-\hat{\vec{a}}_2\right)\right\}\frac{n_1+n_2-2-m}{n_1+n_2-2}-\frac{m}{n_1}-\frac{m}{n_2}$$

as the G_{11}-estimator of the Mahalanobis Distance.

THEOREM 11.1. [Gir54, p.598] *Let* $n_1+n_2-2>m$,

$$\mathbf{E}\,\mu_i=\mathbf{E}\,\nu_i=0,\ \mathbf{E}\,\mu_i^2=\mathbf{E}\,\nu_i^2=1,\ i=1,\dots,m,$$

and for a certain $\beta > 0$

$$\sup_n \max_{i=1,\dots,m} \mathbf{E}\left[|\mu_i|^{4+\beta} + |\nu_i|^{4+\beta}\right] < \infty$$

$$\inf_n \min_{i=1,\dots,m} \lambda_i(R) > 0, \ \sup_n \max_{i=1,\dots,m} \lambda_i(R) < \infty,$$

$$\lim_{n_1,n_2\to\infty} mn_1^{-1} = c_1, \ \lim_{n_1,n_2\to\infty} mn_2^{-1} = c_2, \quad c_1, c_2 < \infty; \quad c_1 + c_2 \neq c_1 c_2,$$

$$\lim_{n_1,n_2\to\infty} (\vec{a}_1 - \vec{a}_2)^T R^{-1} (\vec{a}_1 - \vec{a}_2) \left[n_1^{-1} + n_2^{-1}\right] = 0.$$

Then

$$\underset{n_1,n_2\to\infty}{p\lim} \left\{G_{11} - (\vec{a}_1 - \vec{a}_2)^T R^{-1} (\vec{a}_1 - \vec{a}_2)\right\} = 0.$$

12. G_{12}-REGULARIZED MAHALANOBIS DISTANCE ESTIMATOR

We call

$$\begin{aligned} G_{12} &= \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right)^T \left[\varepsilon I + \varepsilon \theta_{n_1,n_2}^{-1} \hat{R}\right]^{-1} \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right) \\ &\quad - \left[n_1^{-1} + n_2^{-1}\right] \varepsilon \theta_{n_1,n_2}^{-1} \mathrm{Tr}\hat{R} \left[\varepsilon I + \varepsilon \theta_{n_1,n_2}^{-1} \hat{R}\right]^{-1}, \end{aligned} \tag{12.1}$$

the G_{12}-regularized Mahalanobis distance estimator, where $\varepsilon > 0$ is a parameter. Here, θ_{n_1,n_2} is the nonnegative solution of the equation

$$\begin{aligned} &1 - k_{n_1,n_2} + k_{n_1,n_2} \theta_{n_1,n_2}^{-1} \mathrm{Tr}\hat{R} \left[\theta_{n_1,n_2} I + \hat{R}\right]^{-1} = \varepsilon \theta_{n_1,n_2}, \\ &k_{n_1,n_2} = m\left[n_1 + n_2 - 2\right]^{-1}. \end{aligned}$$

It can be seen that there exists a unique nonnegative solution of this equation.

THEOREM 12.1. [Gir54, p.601] *Let the random variables* $\mu_i, \ \nu_i, \ i = 1, \dots, m$ *be independent for every* n,

$$\mathbf{E}\,\mu_i = \mathbf{E}\,\nu_i = 0, \ \mathbf{E}\,\mu_i^2 = \mathbf{E}\,\nu_i^2 = 1, \ i = 1, \dots, m,$$

for a certain $\beta > 0$

$$\sup_n \max_{i=1,\dots,m} \mathbf{E}\left[|\mu_i|^{4+\beta} + |\nu_i|^{4+\beta}\right] < \infty$$

$$\inf_n \min_{i=1,\dots,m} \lambda_i(R) > 0, \quad \sup_n \max_{i=1,\dots,m} \lambda_i(R) < \infty,$$

and the G-condition be satisfied:

$$\limsup_{n_1,n_2\to\infty} m\left[n_1 + n_2 - 2\right]^{-1} < \infty, \quad \limsup_{n_1,n_2\to\infty} \left[n_2 n_1^{-1} + n_1 n_2^{-1}\right] < \infty,$$

$$\sup_{n_1,n_2} (\vec{a}_1 - \vec{a}_2)^T (\vec{a}_1 - \vec{a}_2) < \infty,$$

then, as $\varepsilon > 0$

$$\lim_{n_1,n_2\to\infty} \mathbf{P}\left\{\left[G_{12} - (\vec{a}_1 - \vec{a}_2)^T (\varepsilon I + R)^{-1} (\vec{a}_1 - \vec{a}_2)\right] D_m^{-1/2}\sqrt{n_1+n_2-2} < x\right\}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left\{-y^2/2\right\} \mathrm{d}y,$$

where D_m *are certain constants.*

13. DISCRIMINATION OF TWO POPULATIONS WITH COMMON UNKNOWN COVARIANCE MATRIX. G_{13}-ANDERSON-FISHER STATISTICS ESTIMATOR

Let $\vec{x}_i,\ \vec{y}_j;\ i = 1, ..., n_1;\ j = 1, ..., n_2$ be independent observations of m-dimensional random vectors $\vec{\xi} = \vec{a}_1 + \sqrt{R}\vec{\mu},\ \vec{\mu}^T = \{\mu_i,\ i = 1, \ldots, m\}$ and $\vec{\eta} = \vec{a}_2 + \sqrt{R}\vec{\nu},\ \vec{\nu}^T = \{\nu_i,\ i = 1, \ldots, m\}$ respectively and suppose that random variables $\mu_i,\ \nu_i,\ i = 1, \ldots, m$ are independent for every n.

For the observation classification in the case of two Normal populations, the so-called discriminant function is used:

$$U(\vec{x}) = \left\{\vec{x} - \frac{1}{2}(\vec{\alpha}_1 + \vec{\alpha}_2)\right\}^T R^{-1}(\vec{\alpha}_1 - \vec{\alpha}_2), \vec{x}^T = (x_1, \cdots, x_m).$$

We use empirical mean value vectors and the covariance matrix R,

$$\hat{\vec{a}}_1 = n_1^{-1}\sum_{i=1}^{n_1}\vec{x}_i, \quad \hat{\vec{a}}_2 = n_2^{-1}\sum_{i=1}^{n_2}\vec{y}_i,$$

$$\hat{R} = \frac{1}{n_1+n_2-2}\left\{\sum_{i=1}^{n_1}\left(\vec{x}_i - \hat{\vec{a}}_1\right)\left(\vec{x}_i - \hat{\vec{a}}_1\right)^T + \sum_{i=1}^{n_2}\left(\vec{y}_i - \hat{\vec{a}}_2\right)\left(\vec{y}_i - \hat{\vec{a}}_2\right)^T\right\}.$$

We shall refer to the expression

$$G_{13}(\vec{x}) = \left\{\vec{x} - \frac{1}{2}\left(\hat{\vec{a}}_1 + \hat{\vec{a}}_2\right)^T \hat{R}^{-1}\left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right)\right\}\frac{n_1+n_2-2-m}{n_1+n_2-2}$$

as the $G_{13}(\vec{x})$ -estimator of the discriminant function.

THEOREM 13.1. [Gir54, p.611] *If in addition to the conditions of Theorem 11.1*

$$\lim_{n_1,n_2\to\infty}(\vec{a}_1 + \vec{a}_2)^T R^{-1}(\vec{a}_1 + \vec{a}_2)\left[n_1^{-1} + n_2^{-1}\right] = 0,$$

then

$$\operatorname*{p\lim}_{n\to\infty}\{G_{13}(\nu_i) - U(\nu_i)\} = 0,$$

where $\vec{\nu}_i$ *is an observation of vector* $\vec{\xi}$ *or* $\vec{\eta}$ *which does not depend on* $\vec{x}_i,\ \vec{y}_j;\ i = 1, ..., n_1;$ $j = 1, ..., n_2$ *and distributed as* $N\{\vec{a}_1,\ R\}$ *or* $N\{\vec{a}_2,\ R\}$.

14. G_{14} -ESTIMATOR OF REGULARIZED DISCRIMINANT FUNCTION

If matrix R is singular or ill-conditioned, then instead of the Mahalanobis distance α, its regularized analog is considered

$$\alpha_\varepsilon = (\vec{a}_1 - \vec{a}_2)^T [\varepsilon I + R]^{-1} (\vec{a}_1 - \vec{a}_2), \ \varepsilon > 0.$$

The regularized distance has more useful properties than the distance α. To prove asymptotic normality of G-estimators, it is not necessary that the random vectors $\vec{\xi}$ and $\vec{\eta}$ be normally distributed. As was mentioned in the previous chapters, the estimator

$$G_\varepsilon = \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right)^T \left[\varepsilon I + \hat{R}\right]^{-1} \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right), \ \varepsilon > 0,$$

with empirical mean vectors and the covariance matrix $\hat{\vec{a}}_1, \ \hat{\vec{a}}_2, \ \hat{R}$, is inappropriate for solving multivariate classification problems. Indeed, with the increase of m, the number of components of the vectors $\vec{\xi}$ and $\vec{\eta}$, the number of observations needed for obtaining a given accuracy in the Mahalanobis distance estimation grows rapidly. In this section we assert that under some conditions, there exists an asymptotically Normal G-estimator for the regularized discriminant function, provided that

$$\limsup_{n_1, n_2 \to \infty} \left[m n_1^{-1} + m n_2^{-1}\right] < \infty.$$

Let $\vec{x}_i, \ \vec{y}_j; \ i = 1, \dots, n_1; \ j = 1, \dots, n_2$ be independent observations of m-dimensional independent random vectors $\vec{\xi}$ and $\vec{\eta}$ respectively. We call the expression

$$G_{14}(\vec{x}) = \left\{\vec{x} - \frac{1}{2}\left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right)\right\}^T \left[\varepsilon I + \varepsilon\theta_{n_1,n_2}^{-1}\hat{R}\right]^{-1} \left(\hat{\vec{a}}_1 - \hat{\vec{a}}_2\right)$$
$$- \left[n_1^{-1} + n_2^{-1}\right] \varepsilon\theta_{n_1,n_2}^{-1} \operatorname{Tr} \hat{R} \left[\varepsilon I + \varepsilon\theta_{n_1,n_2}^{-1}\hat{R}\right]^{-1}$$

the G_{14}–estimator for the regularized discriminant function. Here θ_{n_1,n_2} is the nonnegative solution of the equation [Gir54, p.615]

$$1 - k_{n_1,n_2} + k_{n_1,n_2}\theta_{n_1,n_2}^{-1} \operatorname{Tr}\hat{R} \left[\theta_{n_1,n_2} I + \hat{R}\right]^{-1} = \varepsilon\theta_{n_1,n_2},$$
$$k_{n_1,n_2} = m\left[n_1 + n_2 - 2\right]^{-1}$$

It can be seen that there exists a unique nonnegative solution of this equation.

THEOREM 14.1. [Gir54, p.615] *Let the conditions of Theorem 13.1 be satisfied. Then*

$$\lim_{n_1,n_2\to\infty} \max_{i=1,2} \mathbf{P}\left\{\left[G_{14}\left(\vec{\xi}_i\right) - \left\{\vec{\xi}_i - \frac{1}{2}(\vec{a}_1 - \vec{a}_2)\right\}^T (\varepsilon I + R)^{-1} (\vec{a}_1 - \vec{a}_2)\right]\right.$$
$$\left. \times \sqrt{\frac{n_1 + n_2 - 2}{V_m}} < x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-y^2/2\right\} \mathrm{d}y,$$

where $\vec{\xi}_i$ *is an observation which does not depend on* $\vec{x}_i, \ \vec{y}_j; \ i = 1, ..., n_1; \ j = 1, ..., n_2$, *distributed as* $N\{\vec{a}_1, \ R\}$ *or* $N\{\vec{a}_2, \ R\}$ *and* V_m *are certain constants.*

15. G_{15} -ESTIMATOR OF THE NONLINEAR DISCRIMINANT FUNCTION, OBTAINED BY OBSERVATION OF RANDOM VECTORS WITH DIFFERENT COVARIANCE MATRICES

In the case of classifying into two populations based on the Normal distribution, the nonlinear discriminant function is equal to

$$V(\vec{x}) = \frac{1}{2}\left\{-(\vec{x}-\vec{a}_1)^T R_1^{-1}(\vec{x}-\vec{a}_1) + (\vec{x}-\vec{a}_2)^T R_2^{-1}(\vec{x}-\vec{a}_2) - \ln\det R_1 R_2^{-1}\right\}.$$

Let $\vec{x}_i,\ \vec{y}_j;\ i=1,\dots,n_1;\ j=1,\dots,n_2$ be independent observations of m-dimensional random vectors $\vec{\xi}$ and $\vec{\eta}$ respectively; these vectors $\vec{\xi}$ and $\vec{\eta}$ are independent and distributed as $N\{\vec{a}_1,\ R\},\ N\{\vec{a}_2,\ R\}$. As the empirical mean vectors and the covariance matrices R_i, we take:

$$\hat{\vec{a}}_1 = n_1^{-1}\sum_{i=1}^{n_1}\vec{x}_i,\ \hat{\vec{a}}_2 = n_2^{-1}\sum_{i=1}^{n_2}\vec{y}_i,$$

$$\hat{R}_1 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}\left(\vec{x}_i-\hat{\vec{a}}_1\right)\left(\vec{x}_i-\hat{\vec{a}}_1\right)^T,\ \hat{R}_2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}\left(\vec{y}_i-\hat{\vec{a}}_2\right)\left(\vec{y}_i-\hat{\vec{a}}_2\right)^T.$$

We shall refer to the expression

$$\frac{1}{2}\Bigg[-\left(\vec{x}-\hat{\vec{a}}_1\right)^T\hat{R}_1^{-1}\left(\vec{x}-\hat{\vec{a}}_1\right)\frac{n_1-1-m}{n_1-1}+\frac{m}{n_1} + \left(\vec{x}-\hat{\vec{a}}_2\right)^T\hat{R}_2^{-1}\left(\vec{x}-\hat{\vec{a}}_2\right)\frac{n_2-1-m}{n_2-1}+\frac{m}{n_2}\Bigg] - \frac{1}{2}\ln\frac{\det\hat{R}_1\hat{R}_2^{-1}}{\left(1-mn_1^{-1}\right)\left(1-mn_2^{-1}\right)}$$

as the $G_{15}(\vec{x})$ -estimator of the nonlinear discriminant function.

THEOREM 15.1. [Gir54, p.615] *Let random variables* $\mu_i,\ \nu_i,\ i=1,\dots,m$ *be independent for every* n, $\mathbf{E}\,\mu_i = \mathbf{E}\,\nu_i = 0,\quad \mathbf{E}\,\mu_i^2 = \mathbf{E}\,\nu_i^2 = 1,\quad i=1,\dots,m$, *for a certain* $\beta>0$

$$\sup_n \max_{i=1,\dots,m}\mathbf{E}\left[|\mu_i|^{4+\beta}+|\nu_i|^{4+\beta}\right]<\infty,$$

$$\inf_n \min_{i=1,\dots,m}\lambda_i(R)>0,\quad \sup_n \max_{i=1,\dots,m}\lambda_i(R)<\infty,$$

and the G-condition be satisfied:

$$\lim_{m\to\infty} mn_1^{-1}<1,\ \lim_{m\to\infty} mn_2^{-1}<1,$$

$$\sup_m(\vec{a}_1-\vec{a}_2)^T(\vec{a}_1-\vec{a}_2)<\infty.$$

Then,

$$\underset{n\to\infty}{p\lim}\{G_{15}(\nu)-V(\nu)\}=0,$$

where ν is an observation of vector $\vec{\xi}$ or $\vec{\eta}$, which does not depend on $\vec{x}_i$, $\vec{y}_i$.

16. CLASS OF G_{16} - ESTIMATORS IN THE THEORY OF EXPERIMENTAL DESIGN, WHEN THE DESIGN MATRIX IS UNKNOWN

In this section we deal with problems of experimental design under the G-condition

$$\limsup_{n\to\infty} m_n n^{-1}<\infty.$$

Such a condition occurs when the number m of unknown parameters is large, and the number of experiments n has the same order. Given the G-condition, the evaluation of every separate parameter a_i yields under some standard conditions the value $c_1 n^{-1/2}$, where c_1 is some constant. In some cases, the total evaluation error is $c_1 m n^{-1/2}$.

In view of the above, it seems that it is impossible to obtain consistent estimators under the G-condition. However, for many problems it is necessary to evaluate not the parameters a_i, but some function of these parameters $f(a_1,\ldots,a_m)$. But it turns out that in many cases it is possible to find the limit of this function as $n\to\infty$;

$$\limsup_{n\to\infty}|f(\hat{a}_1,\ldots,\hat{a}_m)-g(a_1,\ldots,a_m)|=0.$$

The function g is known and can be obtained as the solution of some equation. This function g differs from the true function f, but when these two functions are known, we can find the G-estimator $G(\hat{a}_1,\ldots,\hat{a}_m)$ of function $f(a_1,\ldots,a_m)$ such that in probability or with probability one, the following limit is valid

$$\limsup_{n\to\infty}|G(\hat{a}_1,\ldots,\hat{a}_m)-f(a_1,\ldots,a_m)|=0.$$

A brief outline of applications of the G-analysis methods described in this section follows.

16.1. G_{16} -estimator of regression models errors. The resolvent method in the theory of experiment design, when the design matrix is random

Let us consider the regression model

$$\vec{y}=X\vec{c}+\vec{\varepsilon},\ \mathbf{E}\,\vec{\varepsilon}=\vec{0},\ \ \mathbf{E}\,\vec{\varepsilon}\vec{\varepsilon}^T=R_{m_n},\ \ \mathbf{E}\,X=A$$

where X is a random matrix, A is a known matrix and the distribution of the matrix X is unknown. Only a simple characteristic of this distribution is known, namely: the entries of the matrix X are independent and their variances are equal to certain constants. Consider a regularized estimator of parameters of this linear regression model

$$\vec{c}_\alpha=\left(\alpha I+X^TX\right)^{-1}X^T\vec{y}. \tag{16.1}$$

Suppose that we have performed experimental design under the random matrix X, which does not depend on the random vector $\vec{\varepsilon}$. Then

$$\vec{c}-\vec{c}_\alpha=\vec{c}-\left\{\alpha I+X^TX\right\}^{-1}X^T\vec{y}.$$

Let $\alpha = 0$, $\mathbf{E}\,\vec{\varepsilon}_i = 0$, $\mathbf{Cov}\,\varepsilon_i\varepsilon_i = n^{-1}\delta_{ij}$ and $p\liminf\limits_{n\to\infty}\lambda_{\min}\left\{X^TX\right\} > 0$. We have

$$\mathbf{E}\,\{\|\vec{c}-\vec{c}_\alpha\|_{\alpha=0}\,|X\} = n^{-1}\mathrm{Tr}\left\{X^TX\right\}^{-1}.$$

Such an expression is inconvenient for finding a minimum on some set of matrices X, since X is a random matrix. It can happen that this matrix will be ill-posed with a positive probability. Therefore it is very important that under general conditions this expression converges to some nonrandom expression, which is more convenient for finding the optimum design. We introduce here the G_{16} estimator of this error $\mathbf{E}\,\{\|\vec{c}-\vec{c}_\alpha\|_{\alpha=0}\,|X\}$:

$$G_{16} = b_n(0),$$

where real analytic function $b_n(\alpha), \alpha > 0$ satisfies the following equation

$$b_n(\alpha) = \frac{1}{n}\mathrm{Tr}\left\{I\left[1+\gamma b_n(\alpha)\right] + (1-\gamma) + \frac{1}{1+\gamma b_n(\alpha)}AA^T\right\}^{-1}, \gamma = \frac{m_n}{n} < 1.$$

We mean here the G_{16}–estimator of the expression for $\mathbf{E}\,\{\|\vec{c}-\vec{c}_0\|\,|X\}$.

THEOREM 16.1. *If for every n the random entries x_{ij} of the matrix X are independent,*

$$\mathbf{E}\,x_{ij} = a_{ij},\quad \mathbf{Var}\,x_{ij} = n^{-1},\quad \limsup_{n\to\infty} m_n^{-1}n < 1,\quad \mathbf{E}\,|(x_{ij}-a_{ij})\sqrt{n}|^{4+\delta} \le c < \infty,\ \delta > 0,$$

$$0 < c_1 \le \lambda_k(AA^T) \le c_2 < \infty,$$

then

$$p\lim_{n\to\infty}\left[\mathbf{E}\,\{\|\vec{c}-\vec{c}_\alpha\|_{\alpha=0}\,|X\} - G_{16}\right] = 0.$$

16.2. G_{16} -estimator of regression models errors. The resolvent method in the theory of experiment design, when the design matrix is an observation of a certain random matrix

Let us consider the regression model

$$\vec{y} = A\vec{c} + \vec{\varepsilon},\ \mathbf{E}\,\vec{\varepsilon} = \vec{0},\quad \mathbf{E}\,\vec{\varepsilon}\vec{\varepsilon}^T = R_{m_n},$$

where A is a matrix.

Here, under the same conditions as in the previous section, we consider the following quality criterion of the least squares estimator

$$\mathbf{E}\,\{\|\vec{c}-\vec{c}_\alpha\|_{\alpha=0}\} = n^{-1}\mathrm{Tr}\left\{A^TA\right\}^{-1}.$$

Suppose that we do not know the matrix A, but we have one observation of the matrix X, where X is a random matrix not depending on random vector $\vec{\varepsilon}$ such that $\mathbf{E}\,X = A$. Then the G_{16} estimator of this error, $n^{-1}\mathrm{Tr}(A^TA)^{-1}$, is equal to:

$$G_{16}(B,C,u+\mathrm{i}v) = \mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-C}^{C}\mathrm{Im}\,G_{27}(z)\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(v-\mathrm{i}u0}\mathrm{d}p,\quad v > 0.$$

Here the G_{27}-estimator of Stieltjes' transform (see Section 27)

$$\varphi\left(z, AA^T\right) = m^{-1}\mathrm{Tr}\,\left[AA^T - zI_{m_n}\right]^{-1}$$

is by definition the following expression:

$$G_{27}\left(\alpha, XX^T\right) = \varphi\left(\hat{\theta}(z), XX^T\right)\left[1 + \gamma\varphi\left(\hat{\theta}(z), XX^T\right)\right]^{-1},$$

where $\hat{\theta}(z)$ is the measurable solution of the G_{27} equation

$$-\hat{\theta}(z)\left\{1 + \frac{1}{n}\mathrm{Tr}\,\left[XX^T - \hat{\theta}(z) I_{m_n}\right]^{-1}\right\}^2 + \left(1 - \frac{m_n}{n}\right)\left\{1 + \frac{1}{n}\mathrm{Tr}\left[XX^T - \hat{\theta}(z) I_{m_n}\right]^{-1}\right\} = -z,$$

$z = t + \mathrm{i}s,\ s > c$ and c is a certain constant.

Theorem 16.2. *If the conditions of Theorem 16.1 are fulfilled, then*

$$\lim_{\nu\downarrow 0}\lim_{B\to\infty}\lim_{C\to\infty} p\lim_{n\to\infty}\left|n^{-1}\mathrm{Tr}(A^T A)^{-1} - G_{16}(B, C, 0 + \mathrm{i}\nu)\right| = 0.$$

16.3. The G_{16} -estimator of regularized regression models errors. The resolvent method in the theory of experiment design, when the design matrix is random

Let us consider the regression model

$$\vec{y} = X\vec{c} + \vec{\varepsilon},\ \mathbf{E}\,\vec{\varepsilon} = \vec{0},\ \mathbf{E}\,\vec{\varepsilon}\vec{\varepsilon}^T = R_{m_n},\ \mathbf{E}\,X = A$$

where X is a random matrix, A is a known matrix and the distribution of the matrix X is unknown. Only simple characteristics of this distribution are known, namely: the entries of the matrix X are independent and their variances are equal to certain constants. Consider a regularized estimator of parameters of this linear regression model

$$\vec{c}_\alpha = \left(\alpha I + X^T X\right)^{-1} X^T\vec{y}. \tag{16.2}$$

Suppose that we have performed experimental design under the random matrix X, which does not depend on random vector $\vec{\varepsilon}$. Then

$$\begin{aligned}\vec{c} - \vec{c}_\alpha &= \vec{c} - \left\{\alpha I + X^T X\right\}^{-1} X^T\vec{y}\\ &= \vec{c} - \left\{\alpha I + X^T X\right\}^{-1} X^T X\vec{c} - \left\{\alpha I + X^T X\right\}^{-1} X^T\vec{\varepsilon}\\ &= \alpha\left\{\alpha I + X^T X\right\}^{-1}\vec{c} - \left\{\alpha I + X^T X\right\}^{-1} X^T\vec{\varepsilon}.\end{aligned}$$

Suppose that the unknown vector $\vec{c}$ satisfies the inequality $\vec{c}^T D\vec{c} \le 1$, where D is a positive defined symmetric matrix. Then we can use the spectral theory of estimation of unknown parameters [Gir84] to find the following regression model error:

$$\max_{\vec{c}^T D\vec{c}\le 1}\mathbf{E}\,(\vec{c} - \vec{c}_\alpha)(\vec{c} - \vec{c}_\alpha) = \alpha\lambda_{\max}\left\{D^{-1/2}\left\{\alpha I + X^T X\right\}^{-2} D^{-1/2}\right\} + \mathrm{Tr}\,\left\{\alpha I + X^T X\right\}^{-2} X^T R X.$$

For simplification we have assumed $D = I$. Now we transform this expression to such a form for which it will be easy to apply the methods of GSA:

$$\max_{\vec{c}^T D\vec{c}\leq 1} \mathbf{E}\,(\vec{c}-\vec{c}_\alpha)(\vec{c}-\vec{c}_\alpha) = \alpha\left\{\alpha+\lambda_{\min}\left[X^T X\right]\right\}^{-2} - \frac{\partial}{\partial\gamma} n^{-1}\mathrm{Tr}\left\{\alpha I + X^T(I+\gamma R)X\right\}^{-1}_{\gamma=0}.$$

For further simplification we will assume that $R = In^{-1}$ and matrix x satisfies the conditions of Theorem 16.1. Then the G_{16} estimator of regression model error is introduced as follows:

$$G_{16} = \alpha\left[\alpha+\beta_1\right]^{-2} + \alpha\frac{\partial b_n(\alpha)}{\partial\alpha} + b_n(\alpha),$$

where

$$\beta_1 = \max\left\{0,\ (1-\gamma)\left[1-\frac{\gamma}{m}\sum_{k=1}^{m}\frac{1}{\alpha_k - v_s}\right] + v_s\left[1-\frac{\gamma}{m}\sum_{k=1}^{m}\frac{1}{\alpha_k - v_s}\right]^2\right\},$$

and v_s are certain real solutions of the L_2 equation (see Chapter 4)

$$1-\sum_{k=1}^{m}\frac{\gamma}{m(\alpha_k - v_i)} = \left[\sum_{k=1}^{m}\frac{\gamma}{m(\alpha_k - v_i)^2}\right]\left\{\frac{1-\gamma}{1-\sum\limits_{k=1}^{m}\frac{\gamma}{m(\alpha_k - v_i)}} + 2v_i\right\},$$

α_k are the eigenvalues of the matrix AA^T and $b_n(\alpha)$ satisfies the following equation

$$b_n(\alpha) = \frac{1}{n}\mathrm{Tr}\left\{I\left[1+\gamma b_n(\alpha)\right] + (1-\gamma) + \frac{1}{1+\gamma b_n(\alpha)}AA^T\right\}^{-1}, \gamma = \frac{m_n}{n} < 1.$$

Under certain conditions the following assertion can be proven:

THEOREM 16.3. *If for every n, random components of the vector $\vec{\varepsilon}^T = \{\varepsilon_1,\ldots,\varepsilon_n\}$ are independent,* $\mathbf{E}\,\varepsilon_i = 0,\ \mathbf{E}\,\varepsilon_i^2 = n^{-1},\ i = 1,\ldots,n,\ 0 < c_1 \leq m_n n^{-1} \leq c_2 < 1,$

$$\alpha_i\left(AA^T\right) \leq c_3,\ i = 1,\ldots,m, D = I,$$

and the entries of the matrix X satisfy the conditions of Theorem 16.1, then

$$\lim_{n\to\infty}\left[G_{16} - \max_{\vec{c}^T\vec{c}\leq 1}\mathbf{E}\,(\vec{c}-\vec{c}_\alpha)^T(\vec{c}-\vec{c}_\alpha)\right] = 0.$$

16.4. G_{16} -estimator of regularized regression models errors. The resolvent method in the theory of experimental design, when the design matrix is a realization of a certain random matrix

Let us consider the regression model

$$\vec{y} = A\vec{c} + \vec{\varepsilon},\ \mathbf{E}\,\vec{\varepsilon} = \vec{0},\ \mathbf{E}\,\vec{\varepsilon}\vec{\varepsilon}^T = R_{m_n}.$$

In this case, in the expression

$$\max_{\vec{c}^T\vec{c}\le 1} \mathbf{E}\,(\vec{c}-\vec{c}_\alpha)(\vec{c}-\vec{c}_\alpha) = \alpha\left\{\alpha + \lambda_{\min}\left[A^TA\right]\right\}^{-2} - \frac{\partial}{\partial\gamma} n^{-1}\mathrm{Tr}\,\left\{\alpha I + A^T(I+\gamma R)A\right\}^{-1}_{\gamma=0}, \alpha > 0. ,$$

a matrix A is unknown, but we know a realization of random matrix $X = A + \Xi$, and we want to estimate this expression for the unknown matrix A. Here the G_{16}-estimator is equal to

$$G_{16} = \alpha\left[\alpha + G_{28}^{\min}\right]^{-2} + \alpha\frac{\partial G_{27}(\alpha)}{\partial\alpha} + G_{27}(\alpha),$$

where the G_{27}-estimator of Stieltjes' transform (see Section 27)

$$\varphi\left(z, AA^T\right) = m^{-1}\mathrm{Tr}\left[AA^T - zI_{m_n}\right]^{-1}$$

is by definition the following expression:

$$G_{27}\left(z, XX^T\right) = \varphi\left(\hat{\theta}(z), XX^T\right)\left[1 + \gamma\varphi\left(\hat{\theta}(z), XX^T\right)\right]^{-1},$$

$\hat{\theta}(z)$ is the measurable solution of the G_{27} equation

$$-\hat{\theta}(z)\left\{1 + \frac{1}{n}\mathrm{Tr}\left[XX^T - \hat{\theta}(z)I_{m_n}\right]^{-1}\right\}^2 + \left(1 - \frac{m_n}{n}\right)\left\{1 + \frac{1}{n}\mathrm{Tr}\left[XX^T - \hat{\theta}(z)I_{m_n}\right]^{-1}\right\} = -z,$$

$G_{28}^{\min}$ is a consistent estimator for minimal eigenvalues $\alpha_m = \lambda_{\min}\left(AA^T\right)$ of the matrix AA^T which equals the minimal measurable solution x of the equation (see Section 28)

$$\lambda_{\min}\left(XX^T\right) = x\left\{1 - \mathrm{Re}\left[\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^{A}\mathrm{Im}\,G_{27}(z)\,\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(x-\mathrm{i}\varepsilon)}\mathrm{d}p\right]\right\}^2.$$

Under certain conditions the following assertion can be proven

THEOREM 16.4. *If for every n, random components of vector $\vec{\varepsilon}^T = \{\varepsilon_1,\ldots,\varepsilon_n\}$ are independent,* $\mathbf{E}\,\varepsilon_i = 0,\ \mathbf{E}\,\varepsilon_i^2 = n^{-1},\ i = 1,\ldots,n,\ 0 < c_1 \le m_n n^{-1} \le c_2 < 1,$

$$\alpha_i\left(AA^T\right) \le c_3,\ i = 1,\ldots,m,$$

and the matrix X satisfies the conditions of Theorem 16.1, then

$$\lim_{n\to\infty}\left[G_{16} - \alpha\left\{\alpha + \lambda_{\min}[A^TA]\right\}^{-2} - \frac{\partial}{\partial\gamma}n^{-1}\mathrm{Tr}\left\{\alpha I + A^{\mathrm{T}}(I+\gamma R)A\right\}^{-1}_{\gamma=0}\right] = 0.$$

17. G_{17}-ESTIMATE OF T^2 -STATISTICS

The multi-dimensional analogue of Student's t^2 statistics is

$$T^2 := \left(\vec{a} - \hat{\vec{a}}\right)^T \hat{R}_{m_n}^{-1} \left(\vec{a} - \hat{\vec{a}}\right)$$

where

$$\hat{R}_{m_n} = (\hat{r}_{ij})_{i,j=1}^{m_n} = n^{-1} \sum_{k=1}^{n} (\vec{x}_k - \hat{\vec{a}})(\vec{x}_k - \hat{\vec{a}})^T, \;\; \hat{\vec{a}} = n^{-1} \sum_{k=1}^{n} \vec{x}_k,$$

$\vec{x}_k, k = 1, ..., n$ are independent observations of a vector $\vec{\xi}$,

$$\mathbf{E}\, \vec{\xi} = \vec{a}, \quad \mathbf{E} \left(\vec{\xi} - \vec{a}\right) \left(\vec{\xi} - \vec{a}\right)^T = R_m.$$

From [Gir69, p.146] we obtain the limit of random variable T^2 when random vectors $\vec{x}_k - \vec{a};\;\; k = 1, \ldots, n$ are independent and G-condition is fulfilled.

THEOREM 17.1. ([Gir69, p.146]*If G-condition* $\limsup_{n\to\infty} mn^{-1} < 1$ *is fulfilled, components* $\xi_{ik},\; i = 1, \ldots, m$ *of the vectors*

$$\vec{\xi}_k = \{\xi_{ik},\; i = 1, \ldots, m\}^T = R_{m_n}^{-1/2} \left[\vec{x}_k - \vec{a}_k\right], \; k = 1, \ldots, n$$

are independent and for some $\delta > 0$

$$\sup_n \max_{\substack{i=1,\ldots,m;\\ k=1,\ldots,n}} \mathbf{E}\, |\xi_{ik}|^{4+\delta} < \infty,$$

$$\vec{a}^T \vec{a} < c_2, \quad \lambda_{\min} \left[R_{m_n}\right] > c_3 > 0,$$

then

$$\operatorname*{plim}_{n\to\infty} \left| \left(1 - mn^{-1}\right) \left(\vec{a} - \hat{\vec{a}}\right)^T \hat{R}_{m_n}^{-1} \left(\vec{a} - \hat{\vec{a}}\right) - mn^{-1} \right| = 0.$$

We call the expression $\left(1 - mn^{-1}\right) \left(\vec{a} - \hat{\vec{a}}\right)^T \hat{R}_{m_n}^{-1} \left(\vec{a} - \hat{\vec{a}}\right) - mn^{-1}$ G_{17}–estimate of T^2-statistics.

18. G_{18}-ESTIMATE OF REGULARIZED T^2 -STATISTICS

A regularized T^2-statistic is defined as follows:

$$T_\varepsilon^2 = n(\vec{a} - \hat{\vec{a}})^T (I\varepsilon + \hat{R}_{m_n})^{-1} (\vec{a} - \hat{\vec{a}}),$$

where $\varepsilon > 0$ is small number.

The G_{18}–estimate of T_ε^2 is equal to

$$G_{18}\left(\varepsilon\right) = mn^{-1} b\left(\varepsilon\right)$$

where $b\left(\varepsilon\right)$ satisfies the equation K_8 (see Chapter 2, Theorem 8.1)

$$b(\varepsilon) = m^{-1} \mathrm{Tr}\, R_m \left\{ I\varepsilon + R_m \left[1 + mn^{-1} b(\varepsilon)\right]^{-1} \right\}^{-1}.$$

THEOREM 18.1. ([Gir69, p.151])*If the conditions of Theorem 17.1 are fulfilled, then*

$$p \lim_{n\to\infty} \left[G_{18}(\varepsilon) - mn^{-1} b(\varepsilon)\right] = 0.$$

The case when $\vec{x} = \hat{\vec{a}}$ is considered in [Gir69, p.204-209].

19. QUASI-INVERSION METHOD FOR SOLVING G-EQUATIONS

Suppose that $f(x)$ is a Borel function in R^{m_n} having partial derivatives of the third order. Let $\vec{x}_1, \ldots, \vec{x}_n$ be independent observations of an m_n-dimensional vector $\vec{\xi}$, $\mathbf{E}\,\vec{\xi} = \vec{a}$. We need a consistent estimator of the value $f(\vec{a})$. Many problems of multivariate statistical analysis can be formulated in these terms. If f is a continuous function we take

$$\hat{\vec{a}} = n^{-1} \sum_{i=1}^{n} \vec{x}_i$$

as the estimator of $\vec{a}$. Then, obviously, for fixed m, $p\lim_{n\to\infty} f(\hat{\vec{a}}) = f(\vec{a})$. But the application of this method in solving practical problems is unsatisfactory due to the fact that the number of observations n necessary to solve the problem with a given accuracy increases sharply with m. It is possible to reduce significantly the number of observations n by making use of the fact that under some conditions, including $\lim_{n\to\infty} mn^{-1} = c$, $0 < c < \infty$, the relation

$$p \lim_{n\to\infty} [f(\hat{\vec{a}}) - \mathbf{E}\, f(\hat{\vec{a}})] = 0 \tag{19.1}$$

holds. We call (19.1) and similar identities the basic relations of the G-analysis of large dimensional observations. The methods of estimating functions of some characteristics of random vectors would be studied by this method.

19.1. G-equations for estimators of differentiable functions of unknown parameters

Suppose that vector $\vec{\xi}$ has a Normal distribution $N(\vec{a}, R_{m_n})$ and consider the functions

$$u(t, \vec{z}) = \mathbf{E}\, f\left(\vec{z} + \vec{a} + \vec{\nu} t^{1/2} n^{-1/2}\right), \tag{19.2}$$

where $t > 0$ is a real parameter, $\vec{z} \in R^{m_n}$, and $\vec{\nu}$ is a Normal $N(0, R_{m_n})$ random vector.

Suppose that the integrals

$$\mathbf{E}\, \frac{\partial^2}{\partial z_i \partial z_j} f\left(\vec{z} + \vec{a} + \vec{\nu} t^{1/2} n^{-1/2}\right)$$

exist. Let us find the differential equation for the function $u(t, \vec{z})$. We note that $\vec{\nu}(t + \Delta t)^{1/2} \approx \vec{\nu} t^{1/2} + \vec{\nu}_1 (\Delta t)^{1/2}$, where $\Delta t \geq 0$, $\vec{\nu}_1$ is a random vector which does not depend on the vector $\vec{\nu}$ and $\vec{\nu} \approx \vec{\nu}_1$. Then

$$\frac{\partial}{\partial t}u\left(t,\vec{z}\right)=\lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\mathbf{E}\left[f\left(\vec{z}+\vec{a}+n^{-1/2}\left(\vec{\nu}t^{1/2}\quad+\vec{\nu}_1(\Delta t)^{1/2}\right)\right)\right.$$
$$\left.-f\left(\vec{z}+\vec{a}+n^{-1/2}\vec{\nu}t^{1/2}\right)\right].$$

Then, by using the expansion of the function f in a Taylor series

$$f\left(\vec{a}+\vec{h}\right)-f(\vec{a})=\sum_{k=0}^{s}\left(\sum_{i=1}^{m_n}\frac{\partial}{\partial a_i}h_i\right)^k f(\vec{a})+o\left(\left\|\vec{h}\right\|\right)$$

we obtain that the functions $u\left(t,\vec{z}\right)$ satisfy the equation

$$\frac{\partial}{\partial t}u\left(t,\vec{z}\right)=Au\left(t,\vec{z}\right);\quad A=\frac{1}{2n}\sum_{i,j=1}^{m_n}r_{ij}\frac{\partial^2}{\partial z_i\partial z_j} \tag{19.3}$$

$$u\left(1,\vec{z}\right)=\mathbf{E}\,f\left(\vec{z}+\hat{\vec{a}}\right),\quad u\left(0,\vec{z}\right)=f\left(\vec{z}+\vec{a}\right),$$

where r_{ij} are the entries of the matrix R_{m_n}. Suppose that the random vector $\vec{\xi}$ has arbitrary distribution with $R_{m_n}=\mathbf{E}\left(\vec{\xi}-\vec{a}\right)\left(\vec{\xi}-\vec{a}\right)^T$. Let

$$\alpha_n\left(kn^{-1},\vec{z}\right)=\mathbf{E}\,f\left\{\vec{z}+\vec{a}+n^{-1}\sum_{p=1}^{k}\left(\vec{x}_p-\mathbf{E}\,\vec{x}_p\right)\right\},$$

$$u_n\left(t,\vec{z}\right)=\alpha_n\left(kn^{-1},\vec{z}\right),\ \ kn^{-1}\le t<(k+1)n^{-1};\ \ k=1,\ldots,n,$$

$$\lim_{n\to\infty}n\mathbf{E}\int_0^1\left(1-t^2\right)\left[\frac{1}{n}\sum_{i=1}^{m_n}\left(\vec{x}_i-\vec{a}_i\right)\left(\frac{\partial}{\partial z_i}\right)\right]^3$$
$$\times f\left\{\vec{z}+\vec{a}+\frac{1}{n}\sum_{i=1}^{k-1}\left(\vec{x}_i-\vec{a}_i\right)+\frac{t}{n}\left(\vec{x}_k-\vec{a}_k\right)\right\}\mathrm{d}t=0.$$

Then, by using the expansion of the function f in a Taylor series, we obtain

$$\begin{aligned}&n\left[\alpha_n\left(\frac{k}{n},\vec{z}\right)-\alpha_n\left(\frac{k-1}{n},\vec{z}\right)\right]\\&=\frac{1}{2n}\sum_{i,j=1}^{m_n}r_{ij}\frac{\partial^2}{\partial z_i\partial z_j}\alpha_n\left(\frac{k-1}{n},\vec{z}\right)+\varepsilon_n,\end{aligned} \tag{19.4}$$

where $\lim_{n\to\infty}\varepsilon_n=0$.

From equation (19.4) we have

$$u_n\left(t,\vec{z}\right)=u_n\left(0,\vec{z}\right)+\int_0^t\frac{1}{2n}\sum_{i,j=1}^{m_n}r_{ij}\frac{\partial^2}{\partial z_i\partial z_j}u_n\left(y,\vec{z}\right)\mathrm{d}y+\varepsilon_n. \tag{19.5}$$

19.2. G-equation of higher orders

Let $f(\vec{x})$, $\vec{x} \in R^{m_n}$ be the Borel function with mixed particular derivatives of order p inclusively; let $\vec{\xi}$, $\mathbf{E}\,\vec{\xi} = \vec{a}$ be a certain m_n-dimensional random vector and let $\vec{x}_1, \ldots, \vec{x}_n$ be independent observations of the vector $\vec{\xi}$.

If, for every $\vec{z} \in R^{m_n}$ and $k = 1, \ldots, n$

$$\lim_{n\to\infty} n\mathbf{E} \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \left(\frac{1}{n} \sum_{i=1}^{m_n} (x_{ik} - a_i) \frac{\partial}{\partial z_i} \right)^p \times f\left(\vec{z} + \vec{a} + \frac{1}{n} \sum_{i=1}^{k-1} (\vec{x}_i - \vec{a}) + \frac{t}{n} (\vec{x}_k - \vec{a}) \right) \mathrm{d}t = 0,$$

$$\sup_{\vec{z} \in R^{m_n}} \mathbf{E} \left| f\left(\vec{z} + \vec{a} + \frac{1}{n} \sum_{i=1}^{k-1} (\vec{x}_i - \vec{a}) \right) \right| < \infty,$$

then

$$\begin{aligned} \varphi_n(t, \vec{z}) &= f(\vec{z} + \vec{a}) + \int_0^t B\varphi_n(y, \vec{z})\, \mathrm{d}y + \varepsilon_n; \\ \varphi_n(1, \vec{z}) &= \mathbf{E} f\left(\vec{z} + \hat{\vec{a}} \right), \end{aligned} \tag{19.6}$$

where

$$\varphi_n(t, \vec{z}) = \mathbf{E} f(\vec{z} + \vec{a} + \vec{\nu}_k), \quad \frac{k}{n} \le t < \frac{k+1}{n}; \quad k = 1, \ldots, n-1,$$

$$\vec{\nu}_k = \frac{1}{n} \sum_{i=1}^{k} (\vec{x}_i - \vec{a}),$$

$$B = \sum_{l=1}^{p-1} \frac{1}{l!} \mathbf{E} \left(\frac{1}{n} \sum_{i=1}^{m_n} (x_{i1} - a_i) \frac{\partial}{\partial z_i} \right)^l.$$

19.3. G-equation for functions of the empirical vector of expectations and the covariance matrix

Let us find the G-equations for the differentiable functions $\varphi_n\left(\hat{\vec{a}}, \hat{R}_{m_n}\right)$ of the empirical vector $\hat{\vec{a}}$ and the covariance matrix $\hat{R}_{m_n}$ which are obtained by independent normally distributed $N(\vec{a}, R_{m_n})$ observations $\vec{x}_1, \ldots, \vec{x}_n$.

Consider the functions

$$u_n(t, \vec{z}, X_{m_n}) = \varphi \left\{ \vec{a} + \vec{z} + R_{m_n}^{1/2} \vec{\eta}_n n^{-1/2}, R_{m_n} + X_{m_n} + R_{m_n}^{1/2} \sum_{s=1}^{k} \left(\frac{1}{n-1} \vec{\eta}_s \vec{\eta}_s^T - I \right) R_{m_n}^{1/2} \right\},$$

where $\vec{\eta}_s$ are independent m_n-dimensional random Normal law $N(0, I)$ vectors , and $X_{m_n} = (x_{ij})$ is a matrix of the parameters of the same order as the matrix R_{m_n}.

If the functions $u_n(t, \vec{z}, X_{m_n})$ can be represented as

$$u_n\left(\frac{k}{n},\vec{z},X_{m_n}\right)-u_n\left(\frac{k-1}{n},\vec{z},X_{m_n}\right)=Au_n\left(\frac{k-1}{n},\vec{z},X_{m_n}\right)+\frac{\varepsilon_n}{n},$$

where

$$A=\frac{1}{2n}\sum_{i,j,p,l=1}^{m_n}\mathbf{E}\left(R_{m_n}^{1/2}\frac{\vec{\eta}_s\vec{\eta}_s^T-I}{n-1}R_{m_n}^{1/2}\right)_{ij}$$
$$\times\left(R_{m_n}^{1/2}\frac{\vec{\eta}_s\vec{\eta}_s^T-I}{n-1}R_{m_n}^{1/2}\right)_{pl}\frac{\partial^2}{\partial x_{ij}\partial x_{pl}}+\frac{1}{2n}\sum_{i,j=1}^{m_n}r_{ij}\frac{\partial^2}{\partial z_i\partial z_j};$$

then we obtain the equation

$$\psi_n(t,\vec{z},X_{m_n})=\varphi(\vec{z}+\vec{a},X_{m_n}+R_{m_n})$$
$$+\int_0^t A\psi_n(y,\vec{z},X_{m_n})\,\mathrm{d}y+\varepsilon_n,$$
$$\psi_n(1,\vec{z},X_{m_n})=\mathbf{E}\,\varphi\left(\vec{z}+\hat{\vec{a}},X_{m_n}+\hat{R}_{m_n}\right)$$

for the functions

$$\psi_n(t,\vec{z},X_{m_n})=u_n\left(\frac{k}{n},\vec{z},X_{m_n}\right);\ \ \frac{k}{n}\le t<\frac{k+1}{n}.$$

19.4. *G*-equation for functions of empirical expectations

Let

$$u_n\left(kn^{-1},\vec{z}\right)=\mathbf{E}\,f\left(\vec{z}+\vec{a}+n^{-1}\sum_{p=1}^{k}(\vec{x}_p-\mathbf{E}\,\vec{x}_p)\right),$$

$$\psi_n(t,\vec{z})=u_n\left(\frac{k}{n},\vec{z}\right),\ \ \frac{k}{n}\le t<\frac{k+1}{n};\ \ k=1,\dots,n.$$

If the limit exists,

$$\lim_{n\to\infty}\left\{n\left[u\left(\frac{k}{n},\vec{z}\right)-u\left(\frac{k-1}{n},\vec{z}\right)\right]-\theta\left(u\left(\frac{k}{n},\vec{z}\right)\right)\right\}=0,$$

where $\theta(y)$ is a certain continuous function on [0,1], then for the functions $\psi_n(t,\vec{z})$ we have

$$\psi_n(t,\vec{z})=\varphi(\vec{z}+\vec{a})+\int_0^t\theta\{\psi_n(y,\vec{z})\}\mathrm{d}y+\varepsilon_n.$$

We deduce the finding of G-estimators of the functions $f(\vec{a})$ to solution of the inverse problem for equation (19.5). The latter consists of finding $\alpha_n(0,z)$ by the function $\alpha_n(1,z)$, which is replaced by the function $f\left(\vec{z}+\hat{\vec{a}}\right)$ based on observations of the random vector $\vec{\xi}$. Of course, the solution of the inverse problem with such a replacement cannot exist in the class of functions $W_2^{(0,2)}$. Therefore, it appears expedient to find a generalized solution of the estimation problem of function $f(\vec{a})$.

Let $\psi(\vec{x})\in L_2$ and let the functional

$$I(\varphi) = \int_D |\alpha_n(1, \vec{x}, \varphi(\cdot)) - \varphi(\vec{x})| \mathrm{d}\vec{x} \tag{19.7}$$

be determined by the functions $\varphi(\vec{x}) \in W_2^{(0,2)}$. Here D is a domain on m-dimensional Euclidean space, which is bounded by the piecewise smooth surface S, and $\alpha_n(1, \vec{x}, \varphi(\cdot))$ is the solution of the equation

$$\alpha_n(t, \vec{x}, \varphi(\cdot)) = \varphi(\vec{x}) + \int_0^t \frac{1}{2n} \sum_{i,j=1}^m r_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \alpha_n(u, \vec{x}, \varphi(\cdot))\, \mathrm{d}u + o(1),$$

at the point $t = 1$. The function $\hat{\varphi}(\vec{x})$ is the solution of the inverse problem if

$$\inf_{\varphi(\cdot) \in W_2^{(0,2)}} I(\varphi) = I(\hat{\varphi}).$$

To solve this problem, we proceed as follows. First, we solve the direct problem

$$\alpha_n(t, \vec{x}, \varphi(\cdot)) = \varphi(\vec{x}) + \int_0^t A\alpha_n(u, \vec{x}, \varphi(\cdot))\, \mathrm{d}u + o(1),$$

where

$$A = \frac{1}{2n} \sum_{i,j=1}^m r_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \;\; \alpha_n(u, \vec{x}, \varphi(\cdot)) = 0, \;\; \in S.$$

Here S is the piecewise smooth boundary of a connected domain D and

$$\alpha_n(1, \vec{x}, \varphi(\cdot)) = \psi(\vec{x})$$

is a given function. Then we have an approximate value for the initial condition of the function $\varphi(x)$. It is quite possible that, in general, such a problem has no solution for the given function. Therefore, it is appropriate to solve the inverse problem approximately with the help of the so-called quasi-inversion method. Thus, we consider the following equation

$$\frac{\partial u(t, \vec{z})}{\partial t} = A_\delta u(t, \vec{z}), \;\; u(1, \vec{z}) = \alpha_n(1, \vec{z}) \tag{19.8}$$

instead of equation (19.5); here A_δ is some operator similar in some sense, to the operator A and such that the solution of equation (19.8) is stable. We can choose

$$A_\delta = A + \delta A^2, \;\; \delta > 0.$$

19.5. Estimator G_{19} of regularized function of unknow parameters

By obtaining the solution of equation (19.6), we can apply the spectral theory of the operator A_δ. Its spectrum is, however, continuous. Therefore, it would be better to replace operator A by an operator A_ε, such that its spectrum is discrete and whose eigenfunctions form the complete orthonormal basis in the Hilbert space L_2. For example, instead of such an operator A, we can choose

$$A_\varepsilon = A + \varepsilon q(\vec{z}) + \delta \left[A + \varepsilon q(\vec{z})\right]^2, \ \ \varepsilon, \delta > 0,$$

where $q(\vec{z})$ is any measurable function such that the operator $A + \varepsilon q(\vec{z}), \ \vec{z} \in R^m$ satisfies the above mentioned condition. From the operator spectral theory, it follows that instead of function $q(\vec{z})$ we can choose any measurable function such that

$$\lim_{\|\vec{z}\| \to \infty} q(\vec{z}) = \infty.$$

Let $\lambda_k(\varepsilon)$ and $\varphi_{k\varepsilon}(\vec{z}), \ \ k = 1, 2, ...$ denote the eigenvalues and eigenfunctions of the operator $A + \varepsilon q(\vec{z}), \ \vec{z} \in R^m$, respectively Now we can give the main form of G_{19}-estimators of function $f(\vec{a})$;

$$\begin{aligned} G_{19} &= \exp\left\{A_\delta - \varepsilon A_\delta^2\right\} f\left(\hat{\vec{\alpha}} + \vec{z}\right)_{\vec{z}=0} \\ &= \sum_{k=0}^{\infty} \exp\left\{\lambda_k(\varepsilon) - \delta\lambda_k^2(\varepsilon)\right\} \int f\left(\hat{\vec{\alpha}} + \vec{z}\right) \varphi_k(\vec{z})\, \mathrm{d}\vec{z} \varphi_k\left(\vec{0}\right), \end{aligned}$$

where

$$A_\varepsilon = \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial z_i \partial z_j} \mathbf{E}\left(\hat{\vec{\alpha}} - \vec{\alpha}\right)_i \left(\hat{\vec{\alpha}} - \vec{\alpha}\right)_j + \varepsilon q(\vec{z}); \quad \varepsilon > 0, \quad \delta > 0,$$

and $q(\vec{z})$ is any continuous function satisfying the condition

$$\liminf_{n \to \infty} \lim_{\|\vec{z}\| \to \infty} q(\vec{z}) = \infty.$$

20. ESTIMATOR G_{20} OF REGULARIZED FUNCTION OF UNKNOWN PARAMETERS

When function $\varphi_n(t, \vec{z})$ satisfies equation (19.6), we use the following operator in equation (19.8),

$$A_\varepsilon = B + \varepsilon q(\vec{z}) + \delta \left[B + \varepsilon q(\vec{z})\right]^2, \ \ \varepsilon, \delta > 0.$$

Here $q(\vec{z})$ is any measurable function such that the operator $B + \varepsilon q(\vec{z}), \ \vec{z} \in R^m$ satisfies the above mentioned condition,

$$B = \sum_{l=1}^{p-1} \frac{1}{l!} \mathbf{E}\left(\frac{1}{n} \sum_{i=1}^{m_n} (x_{i1} - a_i) \frac{\partial}{\partial z_i}\right)^l.$$

From the operator spectral theory, it follows that instead of function $q(\vec{z})$ we can choose any measurable function such that

$$\lim_{\|\vec{z}\|\to\infty} q(\vec{z}) = \infty.$$

Let $\lambda_k(\varepsilon)$ and $\varphi_{k\varepsilon}(\vec{z})$, $k = 1, 2, ...$ denote the eigenvalues and eigenfunctions of the operator $B + \varepsilon q(\vec{z})$, $\vec{z} \in R^m$, respectively. Now we can give the main form of G_{20}-estimators of function $f(\vec{a})$;

$$\begin{aligned} G_{20} &= \exp\{A_\varepsilon\} f\left(\hat{\vec{\alpha}} + \vec{z}\right)_{\vec{z}=0} \\ &= \sum_{k=0}^{\infty} \exp\left\{\lambda_k(\varepsilon) - \delta\lambda_k^2(\varepsilon)\right\} \int f\left(\hat{\vec{\alpha}} + \vec{z}\right)\varphi_k(\vec{z})\, \mathrm{d}\vec{z}\varphi_k(\vec{0}). \end{aligned}$$

21. G_{21}-ESTIMATOR IN THE LIKELIHOOD METHOD

The discussion of this section shows how G-estimators can be constructed from any stochastic experiments. Let $\vec{x}_k$, $k = 1, \dots, n$ be independent observations of vector $\vec{\xi}$ which has a density $p(\vec{\alpha}, \vec{x})$, $\vec{x} = \{x_1, \dots, x_m\}^T$, where $\vec{\alpha} = \{\alpha_1, \dots, \alpha_l\}^T$ is an unknown vector. The likelihood method consists in the following: as an estimator of vector $\vec{\alpha} = \{\alpha_1, \dots, \alpha_l\}^T$, we accept any measurable solution $\hat{\vec{\alpha}}$ of the equation

$$\sup_{\vec{\alpha}\in A} L_n(\vec{\alpha}) = L_n\left(\hat{\vec{\alpha}}\right),$$

where

$$L_n(\vec{\alpha}) = \prod_{k=1}^{n} p(\vec{\alpha}, \vec{x}_k)$$

is the likelihood function. For large dimensional unknown vectors $\vec{\alpha} = \{\alpha_1, \dots, \alpha_l\}^T$, we consider in this section the general likelihood method or G-Method. In this method we make two assumptions: 1. Instead of estimation of vector $\vec{\alpha} = \{\alpha_1, \dots, \alpha_l\}^T$ we consider the estimation problem of density $p(\vec{\alpha}, \vec{x})$, $\vec{x} = \{x_1, \dots, x_m\}^T$ or a certain functional of this density. This assumption is valid in many important practical problems. II. Instead of one sample of observations $\vec{x}_k$, $k = 1, \dots, n$ we consider the scheme of series of samples sequence. The number of unknown parameters m and the number of observations n are related so that the following G-condition is fulfilled:

$$\limsup_{n\to\infty} m_n n^{-1} < \infty.$$

Now we give the main form of G-estimators of density $p(\vec{\alpha}, \vec{x})$:

$$\begin{aligned} G_{21}(\vec{x}) &= \exp\left\{A_\delta - \varepsilon A_\delta^2\right\} p\left(\hat{\vec{\alpha}} + \vec{z}, \vec{x}\right)_{\vec{z}=0} \\ &= \sum_{k=0}^{\infty} \exp\left\{\lambda_k(\varepsilon) - \delta\lambda_k^2(\varepsilon)\right\} \int p\left(\hat{\vec{\alpha}} + \vec{z}, \vec{x}\right)\varphi_k(\vec{z})\, \mathrm{d}\vec{z}\varphi_k(\vec{0}), \end{aligned}$$

where $\hat{\vec{\alpha}}$ is the estimator obtained by the likelihood method, $\lambda_k(\varepsilon)$ and $\varphi_{k\varepsilon}(\vec{z})$, $k = 1, 2, ...$ denote the eigenvalues and eigenfunctions of the operator $A + \varepsilon q(\vec{z})$, $\vec{z} \in R^m$, respectively

$$A_\varepsilon = \frac{1}{2}\sum_{i,j=1}^{m}\frac{\partial^2}{\partial z_i \partial z_j}\mathbf{E}\left(\hat{\vec{\alpha}}-\vec{\alpha}\right)_i\left(\hat{\vec{\alpha}}-\vec{\alpha}\right)_j + \varepsilon q\left(\vec{z}\right);\ \varepsilon > 0,\ \delta > 0,$$

and $q\left(\vec{z}\right)$ is any continuous function satisfying the condition

$$\liminf_{n\to\infty}\lim_{\|\vec{z}\|\to\infty} q\left(\vec{z}\right) = \infty.$$

Here we mention some new properties of the G_{21}-estimators we have thus derived:

I. Under the G-condition and some conditions for the density $p(\vec{\alpha},\vec{x})$, these G_{21}-estimators are consistent and even asymptotically normal.

II. Under the G–condition the standard estimators of the likelihood method lose their asymptotic properties. The G_{21}-estimators of density $p(\vec{\alpha},\vec{x})$ have a complicated form, but for some cases it can be see that a new estimator of $\vec{\alpha}$ in the expression for the G_{21}–estimator depends on vector $\vec{x} = \{x_1,\dots,x_m\}^T$.

Let us consider one example. Because it is very difficult to find a simple evident expression for the G_{21}-estimator, let

$$p\left(\vec{\alpha},\vec{x}\right) = (2\pi)^{-m_n/2}\exp\left\{-\left\|\vec{\alpha}-\vec{x}\right\|^2/2\right\}, \vec{\alpha}\in R^{m_n}, \vec{x}\in R^p.$$

Then the likelihood estimator of vector $\vec{\alpha}$ is equal to

$$\hat{\vec{\alpha}} = n^{-1}\sum_{k=1}^{n}\vec{x}_k.$$

Putting this estimator in the density $p\left(\vec{\alpha},\vec{x}\right)$, we get

$$p\left(\hat{\vec{\alpha}},\vec{x}\right) = (2\pi)^{-m_n/2}\exp\left\{-\frac{\left\|\vec{\alpha}-\vec{x}\right\|^2}{2} + \frac{\left(\vec{\alpha}-\vec{x}\right)^T\vec{\eta}}{\sqrt{n}} - \frac{\left\|\vec{\eta}\right\|^2}{2n}\right\},$$

where $\vec{\eta} = \left(\hat{\vec{\alpha}}-\vec{\alpha}\right)\sqrt{n}$.

Obviously, if

$$\lim_{n\to\infty} m_n n^{-1} = c,\ 0 < c < \infty;\ \limsup_{n\to\infty}\left\|\vec{\alpha}-\vec{x}\right\|^2 n^{-1} = 0, \tag{21.1}$$

then

$$\lim_{n\to\infty}\mathbf{E}\left[\frac{p\left(\hat{\vec{\alpha}},\vec{x}\right)}{p\left(\vec{\alpha},\vec{x}\right)} - 1\right]^2 = \left[\mathrm{e}^{-c/2}-1\right]^2 > 0.$$

But if we take the G_{21}-estimator

$$G\left(\hat{\vec{\alpha}},\vec{x}\right) = p\left(\hat{\vec{\alpha}},\vec{x}\right)\mathrm{e}^{c/2}$$

of the density $p\left(\vec{\alpha},\vec{x}\right)$, we will have under the same condition (21.1):

$$\lim_{n\to\infty} \mathbf{E}\left[\frac{G_{21}\left(\hat{\vec{\alpha}}, \vec{x}\right)}{p\left(\vec{\alpha}, \vec{x}\right)} - 1\right]^2 = 0.$$

It is easy to see that the G_{21}-estimator is much better than the standard likelihood estimator.

22. G_{22}-ESTIMATOR IN THE CLASSIFICATION METHOD

Certain problems of classification can be formulated in the following way: let Borel functions $f_n\left(x_1, \ldots, x_n, \theta\right)$ of sample observations $x_1, \ldots, x_n$ and unknown parameter θ be given. We need conditions on the function $f_n\left(x_1, \ldots, x_n, \theta\right)$ and the distribution of $x_1, \ldots, x_n$ such that the measurable solution θ of equations

$$f_n\left(x_1, \ldots, x_n, \theta\right) = 0; \quad \text{or} \quad \sup_\theta f_n\left(x_1, \ldots, x_n, \theta\right) = f_n\left(x_1, \ldots, x_n, \hat{\theta}\right)$$

converges in probability to the corresponding solution θ of equations

$$\mathbf{E}\, f_n\left(x_1, \ldots, x_n, \theta\right) = 0; \quad \text{or} \quad \sup_\theta \mathbf{E}\, f_n\left(x_1, \ldots, x_n, \theta\right) = \mathbf{E}\, f_n\left(x_1, \ldots, x_n, \hat{\theta}\right).$$

If the densities $p\left(x, \theta\right)$ of the random variables $x_1, \ldots, x_n$ exist and

$$f_n\left(x_1, \ldots, x_n, \theta\right) = \sum_{k=1}^{n} \ln p\left(x_k, \theta\right)$$

then this problem is known as the problem of the method of the maximum likelihood method.

22.1. Integral representation method. Limit theorems of the type of the law of large numbers

One of the important problems of classification theory is the problem of finding

$$\sup_{\vec{\alpha}\in A} \int f\left(\vec{\alpha}, \vec{x}\right) \mathrm{d}F\left(\vec{x}\right),$$

where $f\left(\vec{\alpha}, \vec{x}\right),\ \vec{x} \in R^m,\ \vec{\alpha} \in R^s$ is some measurable function, $F\left(\vec{x}\right)$ is a multivariate distribution function and A is some measurable set. Suppose that the point $\vec{\alpha}_0$, at which the function $\int f\left(\vec{\alpha}, \vec{x}\right) \mathrm{d}F\left(\vec{x}\right)$ attains its maximal value, is unique. This proposition is equivalent to the equation,

$$\sup_{\vec{\alpha}\in A} \int f\left(\vec{\alpha}, \vec{x}\right) \mathrm{d}F\left(\vec{x}\right) = \int f\left(\vec{\alpha}_0, \vec{x}\right) \mathrm{d}F\left(\vec{x}\right), \tag{22.1}$$

having a unique solution. In many problems the function $F\left(\vec{x}\right)$ is unknown and one uses an empirical distribution function obtained from n independent observations $\vec{x}_1, \ldots, \vec{x}_n$ of the random vector $\vec{\xi}$. Then, instead of equation (22.1), we get the equation

$$\sup_{\vec{\alpha}\in A} n^{-1} \sum_{k=1}^{n} f\left(\vec{\alpha}, \vec{x}_k\right) = n^{-1} \sum_{k=1}^{n} f\left(\hat{\vec{\alpha}}, \vec{x}_k\right). \tag{22.2}$$

Consequently, the problem is reduced to finding the extreme of the empirical function and conditions when $p\lim_{n\to\infty} \hat{\vec{\alpha}} = \vec{\alpha}_0$, and proving that the asymptotic distribution of the vectors $(\hat{\vec{\alpha}} - \vec{\alpha}_0)c_n$, under some normalization, is normal. The integral representation method of the proof of limit theorems for the extremum of empirical functions is based on the proposition that the function $f(\vec{\alpha}, \vec{x})$ can be represented in the following form

$$f(\vec{\alpha}, \vec{x}) = \int \exp\left(i\vec{\alpha}^T \vec{y}\right) p(\vec{y}, \vec{x})\, d\vec{y}; \quad \vec{y} \in R^s, \tag{22.3}$$

$$f(\vec{\alpha}, \vec{x}) = \int \exp\left(i\vec{x}^T \vec{y}\right) q(\vec{y}, \vec{\alpha})\, d\vec{y}; \quad \vec{y} \in R^m, \tag{22.4}$$

where p and q are absolutely integrable functions, or in the form of convergent series

$$f(\vec{\alpha}, \vec{x}) = \sum_{k=1}^{\infty} c_k(\vec{\alpha})\, \varphi_k(\vec{x}), \quad f(\vec{\alpha}, \vec{x}) = \sum_{k=1}^{\infty} d_k(\vec{x})\, \psi_k(\vec{\alpha}), \tag{22.5}$$

where φ_k, ψ_k are some sequences of orthonormalized systems of functions:

$$c_k(\vec{\alpha}) = \int f(\vec{\alpha}, \vec{y})\varphi_k(\vec{y})\, d\vec{y}; \;\; d_k(\vec{x}) = \int f(\vec{\alpha}, \vec{x})\psi_k(\vec{\alpha})\, d\vec{\alpha}\,.$$

We note that sometimes a function f can also be represented in the form of a Stieltjes' integral

$$f(\vec{\alpha}, \vec{x}) = \int \exp\left(i\vec{\alpha}^T \vec{y}\right) dG(\vec{y}, \vec{x}); \quad \vec{y} \in R^s;$$
$$f(\vec{\alpha}, \vec{x}) = \int \exp\left(i\vec{x}^T \vec{y}\right) dK(\vec{y}, \vec{\alpha})\, d\vec{y}; \quad \vec{y} \in R^m,$$

where G and K are functions of bounded variation. Using, for example, formula (22.4) we can reduce equation (22.2) to the form

$$\sup_{\vec{\alpha}\in A} n^{-1} \sum_{k=1}^{n} f(\vec{\alpha}, \vec{x}_k) = \int q(\vec{y}, \vec{\alpha}) \left\{ n^{-1} \sum_{k=1}^{n} \exp\left(i\vec{x}_k^T \vec{y}\right) \right\} d\vec{y}. \tag{22.6}$$

On the right-hand side of this equality we have the sum of independent random variables, for which the limit theorems can be used. On the basis of this equation the following assertion is proved in [Gir44, p. 133–141].

THEOREM 22.1. *Let $\vec{x}_1, \ldots, \vec{x}_n$ be independent observations of a random vector $\vec{\xi}$, and suppose that a function f is represented in the form of the integral (22.4), equation (22.1) has a unique bounded solution $\vec{\alpha}_0$, and*

$$\int \sup_{\vec{\alpha}\in A} |q(\vec{y}, \vec{\alpha})| \left[1 - \left|\mathbf{E} \exp\left(i\vec{x}_1^T \vec{y}\right)\right|^2\right]^{1/2} d\vec{y} < \infty.$$

Then $p\lim_{n\to\infty} \hat{\vec{\alpha}}_n = \vec{\alpha}_0$, where $\hat{\vec{\alpha}}_n$ is any measurable solution of equation (22.2).

Following the main principles of GSA, we introduce here the G_{22} estimator of function $f(\vec{\alpha}) = \int p(\vec{\alpha}_n, \vec{x})\, d\hat{F}(\vec{x})$:

$$G_{22} = \left[\exp\left\{A_\varepsilon + \delta A_\varepsilon^2\right\}\int p\left(\hat{\vec{\alpha}}_n + \vec{z}, \vec{x}\right)\mathrm{d}\hat{F}\left(\vec{x}\right)\right]_{\vec{z}=\vec{0}},$$

where

$$A_\varepsilon = \frac{1}{2}\sum_{i,j=1}^{m}\frac{\partial^2}{\partial z_i \partial z_j}\mathbf{E}\left(\hat{\vec{\alpha}} - \vec{\alpha}\right)_i\left(\hat{\vec{\alpha}} - \vec{\alpha}\right)_j + \varepsilon q\left(\vec{z}\right);\ \varepsilon > 0,\ \delta > 0,$$

and $q\left(\vec{z}\right)$ is any continuous function satisfying the condition

$$\liminf_{n\to\infty}\lim_{\|\vec{z}\|\to\infty} q\left(\vec{z}\right) = \infty.$$

23. G_{23}-ESTIMATOR IN THE METHOD OF STOCHASTIC APPROXIMATION

The problem of stochastic approximation can be formulated in the following way: Suppose that in the space R^{m_n} there are defined functions $\vec{f}_k\left(\vec{x}\right),\ \vec{x} \in R^{m_n},\ \vec{f}\left(\vec{x}\right) = \left\{f_k\left(\vec{x}\right),\ k = 1, \ldots, q_n\right\}^T$. The goal is to find the solution of the system of equations $\vec{f}\left(\vec{x}\right) = \vec{0}$. Suppose that in every point $\vec{x} \in D \in R^{m_n}$ we can calculate vector function $\vec{f}\left(\vec{x}\right)$ with certain random errors. Observations $\vec{\xi}\left(\vec{x}_s\right),\ s = 1, 2\ldots;\ \vec{\xi}\left(\vec{x}_s\right) = \left\{\xi_k\left(\vec{x}_s\right),\ k = 1, \ldots, q_n\right\}^T$ of vectors $\vec{f}\left(\vec{x}_s\right) + \vec{\eta}_s,\ s = 1, 2\ldots,$ where $\vec{\eta}_s,\ s = 1, 2, \ldots$ are certain random vectors, are available. It is necessary with the help of these observations to find a sequence of the points converging on probability to the solution of the system of equation $\vec{f}\left(\vec{x}\right) = \vec{0}$.

According to the Robbins-Monro procedure we choose the following sequence of the points $\vec{x}_s \in R^{m_n}$ satisfying the system of recurrence equations:

$$\left(\vec{x}_{s+1} - \vec{x}_s\right)\alpha_s^{-1} = \vec{\xi}\left(\vec{x}_s\right),\ \vec{x}_1 \in D \in R^{m_n}, \tag{23.1}$$

where α_s is a certain sequence of numbers.

Let us estimate the rate of convergence of the sequence $\vec{x}_{p+1}$ to the solution of the system of equations $\vec{f}\left(\vec{x}\right) = \vec{0}$. Suppose that there exists the unique solution $\vec{x}_0$ of this system of equations and that in the neighborhood of this solution

$$\left(\vec{x}_{p+1} - \vec{x}_p\right) = \left(\vec{x}_p - \vec{x}_0\right) + \alpha_p \Xi_p\left(\vec{x} - \vec{x}_0\right). \tag{23.2}$$

where Ξ_k are random matrices, $\mathbf{E}\,\Xi_k = B$.

Using (23.2) and (23.1) we get

Denote $\vec{y}_{p+1} = \left(\vec{x}_{p+1} - \vec{x}_p\right)\gamma_{p+1}$, where γ_{p+1} is a normalizing sequence. Then

$$\vec{y}_{p+1} = \gamma_{p+1}\gamma_p^{-1}\left[I_{m_n} + \alpha_p \Xi_p\right]\vec{y}_p = \vec{y}_0\prod_{s=0}^{p}\left\{\gamma_{s+1}\gamma_s^{-1}\left[I_{m_n} + \alpha_s \Xi_s\right]\right\}.$$

We can now use for this product of random matrices the G_{42}-estimator when $mp^{-1} \to c$ (see Section 42). The obtained estimator will be denoted as G_{23}.

24. CLASS OF ESTIMATORS G_{24}, WHICH MINIMIZES CERTAIN MEAN-SQUARE RISKS

Let $\vec{\xi}^T = \{\xi_1, \dots, \xi_{m_n}\}$ be a random Normal $N\left(\vec{a}, \sigma^2 I_{m_n}\right)$ vector; assume that $\vec{x}_s$; $s = 1, 2, \dots, n$ are independent observations of vector $\vec{\xi}$. Obviously

$$\hat{\vec{a}} = n^{-1} \sum_{k=1}^{n} \vec{x}_k$$

is an unbiased estimator of vector $\vec{a}$ and

$$\mathbf{E}\left(\hat{\vec{a}} - \vec{a}\right)^T \left(\hat{\vec{a}} - \vec{a}\right) = \sigma^2 m_n n^{-1}.$$

C. Stein [St1–3] has proposed a method to find estimators $\tilde{\vec{a}}$ such that

$$\mathbf{E}\left(\tilde{\vec{a}} - \vec{a}\right)^T \left(\tilde{\vec{a}} - \vec{a}\right) < \sigma^2 m_n n^{-1}.$$

His method consists in the following: let us consider estimators

$$\tilde{\vec{a}} = \hat{\vec{a}} + n^{-1} \vec{g}\left(\hat{\vec{a}}\right),$$

where $\vec{x} \in R^{m_n}$, $\vec{g}(\vec{x}) \in R^{m_n}$. It is easy to verify that

$$\begin{aligned}
&\mathbf{E}\left(\hat{\vec{a}} - \vec{a}\right)^T \left(\hat{\vec{a}} - \vec{a}\right) - \mathbf{E}\left\{\hat{\vec{a}} + n^{-1}\vec{g}\left(\hat{\vec{a}}\right) - \vec{a}\right\}^T \left\{\hat{\vec{a}} + n^{-1}\vec{g}\left(\hat{\vec{a}}\right) - \vec{a}\right\} \\
&= -\frac{2}{n}\mathbf{E}\left(\hat{\vec{a}} - \vec{a}\right)^T \vec{g}\left(\hat{\vec{a}}\right) - n^{-2}\mathbf{E}\,\vec{g}^T\left(\hat{\vec{a}}\right)\vec{g}\left(\hat{\vec{a}}\right).
\end{aligned}$$

Using this equality we have

$$\begin{aligned}
&\mathbf{E}\left(\hat{\vec{a}} - \vec{a}\right)^T \left(\hat{\vec{a}} - \vec{a}\right) - \mathbf{E}\left\{\tilde{\vec{a}} - \vec{a}\right\}^T \left\{\tilde{\vec{a}} - \vec{a}\right\} \\
&= \frac{2}{n^2}\mathbf{E}\sum_{i=1}^{m_n} \frac{\partial g\left(\hat{\vec{a}}\right)}{\partial x_i} - n^{-2}\mathbf{E}\,\vec{g}^T\left(\hat{\vec{a}}\right)\vec{g}\left(\hat{\vec{a}}\right).
\end{aligned} \tag{24.1}$$

Suppose that function $\vec{g}\left(\hat{\vec{a}}\right)$ is equal to

$$\vec{g}(\vec{x}) = \text{grad}\left[\ln\varphi(\vec{x})\right],$$

where $\varphi(\vec{x})$, $\vec{x} \in R^{m_n}$ is a twice differentiable function. For this function the following equality is valid

$$\sum_{i=1}^{m_n} \frac{\partial g_i(\vec{x})}{\partial x_i} = \sum_{i=1}^{m_n} \frac{\partial}{\partial x_i}\left[\frac{1}{\varphi(\vec{x})}\frac{\partial\varphi(\vec{x})}{\partial x_i}\right] = -\vec{g}^T(\vec{x})\,\vec{g}(\vec{x}) + \varphi(\vec{x})\,\Delta\varphi(\vec{x}).$$

Here

$$\Delta = \sum_{i=1}^{m_n} \frac{\partial^2}{\partial x_i^2}$$

is the Laplace operator. Using this equality and (24.1) we get

$$\begin{aligned}&\mathbf{E}\left(\hat{\vec{a}}-\vec{a}\right)^T\left(\hat{\vec{a}}-\vec{a}\right)-\mathbf{E}\left\{\tilde{\vec{a}}-\vec{a}\right\}^T\left\{\tilde{\vec{a}}-\vec{a}\right\}\\&=n^{-2}\mathbf{E}\,\vec{g}^T\left(\hat{\vec{a}}\right)\vec{g}\left(\hat{\vec{a}}\right)-n^{-2}\mathbf{E}\left[\varphi\left(\hat{\vec{a}}\right)\right]^{-1}\Delta\varphi\left(\hat{\vec{a}}\right).\end{aligned}$$

The right part of this equality is positive if $\Delta\varphi\left(\hat{\vec{a}}\right)\leq 0$. Such functions are called superharmonic functions and do not exist on the real line or on the complex plane. Therefore, in these two cases the Stein method does not give improvement of the estimator $\hat{\vec{a}}$. But if $m_n\geq 3$, then such superharmonic functions exist.

In G-analysis one of the axioms requires that we estimate some functions of parameters with unknown distribution. Nevertheless, using the G-analysis methods, we can try to find some estimators which minimize certain risk functions. We will denote such estimators by the symbol G_{24}. Consider several examples:

24.1. The risk function of the estimator of inverse covariance matrix R^{-1}

The risk function of the estimator of inverse covariance matrix R^{-1} is equal to

$$m_n^{-1}\mathbf{E}\,\mathrm{Tr}\left\{R_{m_n}^{-1}-G\right\}^2,$$

where estimator G is a certain matrix function of observations $\vec{x}_1,\dots,\vec{x}_n$ of a random vector with the covariance matrix R_{m_n}. In the Section 3 it was proved that we can consider the estimator $G_3=\hat{R}_{m_n}^{-1}\left[1-m_nn^{-1}\right]$, where $\hat{R}_{m_n}$ is the standard empirical covariance matrix. For such estimator and the standard estimator under certain conditions

$$m_n^{-1}\mathbf{E}\,\mathrm{Tr}\left\{R_{m_n}^{-1}-G_3\right\}^2<m_n^{-1}\mathbf{E}\,\mathrm{Tr}\left\{R_{m_n}^{-1}-\hat{R}_{m_n}^{-1}\right\}^2.$$

24.2. The Stein's risk function

We can use G–estimators G_1 and G_3 for the minimization of the Stein's risk function

$$\mathrm{Tr}\,R_{m_n}^{-1}G-\ln\det R_{m_n}^{-1}G+m_n.$$

The obtained estimator we call the G_{24}–estimator.

25. G_{25}- ESTIMATOR OF THE STIELTJES TRANSFORM OF THE PRINCIPAL COMPONENTS

Let $\lambda_k,\ \vec{\varphi}_k;\ k=1,\dots,m$ be eigenvalues and corresponding eigenvectors of the covariance matrix R_m. Consider the spectral function

$$\nu_m(x)=\sum_{k=1}^{m}\left(\vec{a}^T\vec{\varphi}_k\right)\left(\vec{b}^T\vec{\varphi}_k\right)\chi\left(\lambda_k<x\right),$$

where $\vec{a},\ \vec{b}$ are m-dimensional vectors. We call $\vec{x}^T\vec{\varphi}_k$ a k-th principal value. Here $\vec{x}$ is an observation of a random vector with the covariance matrix R. Consider Stieltjes' transform of the spectral function $\nu_m(x)$:

$$\int_0^\infty(1+tx)\mathrm{d}\nu_m(x)=\vec{a}^T\left(I+tR_{m_n}\right)^{-1}\vec{b}.$$

The G_{25} estimator for the quadratic forms of resolvents of covariance matrices $\vec{a}^T\left(I+t\hat{R}_{m_n}\right)^{-1}\vec{b}$ is equal to: $G_{25}=\vec{a}^T\left(I+\theta\hat{R}_{m_n}\right)^{-1}\vec{b}$, where θ is the positive solution of the equation

$$\theta\left[1-m_n n^{-1}+n^{-1}\mathrm{Tr}\left(I+\theta\hat{R}_{m_n}\right)^{-1}\right]=t;\ \ t>0.$$

26. G_{26} - ESTIMATOR OF EIGENVALUES OF THE COVARIANCE MATRIX

Let $\mu_{m_n}\leq\ldots\leq\mu_1$ be the eigenvalues, let $\vec{h}_i, i=1,\ldots,m_n;\ \ h_{1i}>0$ be the eigenvectors of the covariance matrix R_{m_n} and let the vectors $\vec{x}_1,\ldots,\vec{x}_n$ be observations of a random vector $\vec{\xi}$ distributed according to the Normal law $N\left(\vec{a},\ R_{m_n}\right)$. Here $\hat{R}_{m_n}$ is the empirical covariance matrix

$$\hat{R}_{m_n}=(n-1)^{-1}\sum_{k=1}^{n}(x_k-\hat{a})(x_k-\hat{a})^T$$

and $\hat{a}$ is the empirical expectation:

$$\hat{a}=n^{-1}\sum_{k=1}^{n}x_k.$$

It is well known that the maximal likelihood estimators of the simple eigenvalues $\mu_{m_n}<...<\mu_1$ and corresponding orthonormal eigenvectors $\vec{h}_i, i=1,\ldots,m_n;\ \ h_{1i}>0$ of the matrix R_{m_n} and of the vector $\vec{a}$ obtained from independent vectors $\vec{x}_1,\ldots,\vec{x}_n$ which are observations of a random Normal $N\left(\vec{a},R_{m_n}\right)$ vector $\vec{\xi}$ are equal to ([And1])

$$\mu_{m_n}\left(\hat{R}_{m_n}\right)\leq...\leq\mu_1\left(\hat{R}_{m_n}\right),\ h_i\left(\hat{R}_{m_n}\right),\ \ \hat{\vec{a}},\ \ i=1,...,m_n.$$

In view of the optimal properties of these estimates, it seems that the problem of asymptotic estimation of the simple eigenvalues $\mu_{m_n}<...<\mu_1$ and the eigenvectors

$$\vec{h}_i, i=1,\ldots,m_n;\ \ h_{1i}>0$$

has been solved. And this is indeed so, if m is small and does not depend on n . For large values of m, one can try to reduce the bias of these estimates by using the "jackknife method" or the "bootstrap method". These methods do not give a clear analytic formula, although they involve a large number of computer calculations. The question arises as to whether all the estimates of the eigenvalues μ_i are necessary to the solution of practical problems. An analysis of many problems suggests that they are not all necessary. But then one can try to use G-analysis methods [Gir40, Gir41, Gir50, Gir51, Gir53, Gir55, Gir57, Gir58] to find consistent estimates of a finite number of eigenvalues μ_i that do not depend on n under the condition

$$\limsup_{n\to\infty} m_n n^{-1}<1.$$

26.1. The criticism of the multivariate statistical analysis. G_2-estimator for Stieltjes' transform of the normalized spectral function of covariance matrices

In this section we consider an important problem of G-analysis, namely, that of estimation of the eigenvalues of a covariance matrix. In spite of many studies of eigenvalues of random matrices we do not have good estimators for them. Indeed, we do not have any information on eigenvalues. Therefore, it is very difficult to apply the perturbation formulas to find these estimators. Let R_m be a covariance matrix, $\lambda_k;\ k=1,\dots,m$ and let $\vec{\eta}_k;\ k=1,\dots,m$ be its eigenvalues and the corresponding eigenvectors, chosen in such a way that they are random variables. We consider a new spectral function

$$\nu_n\left(x, R_{m_n}\right)=\sum_{k=1}^{m} \vec{a}^T \vec{\eta}_k \vec{\eta}_k^T \vec{b} \chi\left(\lambda_k<x\right),$$

where $\vec{a}$ and $\vec{b}$ are any nonrandom real vectors of dimension m. Such a spectral function contains all information about eigenvalues and eigenvectors. With its help, we can find new (more precise) estimators for the principal values and the eigenvalues of the covariance matrix R_m by the independent observations of the random vector $\vec{\xi}$. Note that with the help of this estimator we can significantly decrease the number of observations required to solve practical estimation problems of principal values for large values of m. The criticism of the multivariate statistical analysis of large-dimensional observations was due to the fact that the error of estimates in this analysis usually is $mn^{-1/2}$ or $\sqrt{m}n^{-1/2}$, where m is the number of parameters to be estimated, and n is the number of observations. It is evident that the number of observations needed for estimation with given accuracy increases sharply with the growth of m. Hence, multivariate statistical analysis does not help us to solve practical problems involving observations on large-dimensional vectors. After many years of research, it appears that under the G-condition

$$\limsup_{n\to\infty} m_n n^{-1}<\infty,$$

consistent and asymptotically normal estimates of many functions $\varphi\left(R_{m_n}\right)$ do not exist. However, by the developed matrix spectral theory, we can establish that they do exist under the G- condition

$$\limsup_{n\to\infty} m_n n^{-1}<\infty,\ \ p\lim_{n\to\infty}\left[\varphi\left(\hat{R}_{m_n}\right)-\psi\left(R_{m_n}\right)\right]=0,$$

where ψ is some known measurable function of the matrix R_{m_n} entries. This equation will be called the G-equation. This is the principal statement making up the basis of the G-analysis of observations of large dimensions. We must recall that the G_2-consistent estimator for the trace of resolvents of covariance matrices

$$m_n^{-1}\operatorname{Tr}\left(\hat{R}_{m_n}-zI_{m_n}\right)^{-1},\ z=t+\mathrm{i}s,\ s>0$$

has the form:

$$G_2(z)=z^{-1}\hat{\theta}(z)\, m_n^{-1}\operatorname{Tr}\left\{\hat{R}_{m_n}-\hat{\theta}(z)\, I_{m_n}\right\}^{-1},$$

where $\hat{\theta}(z)$ is the measurable solution of the equation

$$\hat{\theta}(z)\frac{1}{n}\mathrm{Tr}\left\{\hat{R}_{m_n}-\hat{\theta}(z)I_{m_n}\right\}^{-1}-\left(1-\frac{m_n}{n}\right)+\frac{\hat{\theta}(z)}{z}=0.$$

The solution of the problem of finding the G_{26}-estimator of the eigenvalues λ_i turned out to be extremely complex. For example, the $G_{26}^{\max}(A,B,\varepsilon)$ -consistent estimator for the maximal eigenvalue $\lambda_1(R_m)$ of covariance matrix R_m is equal to a maximal measurable solution x of the equation

$$\lambda_1\left(\hat{R}_{m_n}\right)=x\mathrm{Re}\left\{1-\gamma-\gamma x\left[\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^A\mathrm{Im}\,G_2(z)\,\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(x-\mathrm{i}\varepsilon)}\mathrm{d}p\right]\right\},$$

where $\gamma=m_nn^{-1}$,

$$G_2(z)=z^{-1}\hat{\theta}(z)\,m_n^{-1}\mathrm{Tr}\left\{\hat{R}_{m_n}-\hat{\theta}(z)I_{m_n}\right\}^{-1},$$

$z=t+\mathrm{i}s,\ s>c>0,\ c$ is a certain constant, $\hat{\theta}(z)$ is the measurable solution of the equation

$$\hat{\theta}(z)\frac{1}{n}\mathrm{Tr}\left\{\hat{R}_{m_n}-\hat{\theta}(z)I_{m_n}\right\}^{-1}-\left(1-\frac{m_n}{n}\right)+\frac{\hat{\theta}(z)}{z}=0.$$

Similarly we defined the $G_{26}^{\min}(A,B,\varepsilon)$ -consistent estimator for minimal eigenvalue $\lambda_m(R_m)$ of covariance matrix R_{m_n}, which is equal to a minimal measurable solution x of the equation

$$\lambda_{m_n}\left(\hat{R}_{m_n}\right)=x\left\{1-\gamma-\gamma x\mathrm{Re}\left[\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^A\mathrm{Im}\,G_2(z)\,\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(x-\mathrm{i}\varepsilon)}\mathrm{d}p\right]\right\},$$

where

$$G_2(z)=z^{-1}\hat{\theta}(z)\,m_n^{-1}\mathrm{Tr}\left\{\hat{R}_{m_n}-\hat{\theta}(z)I_{m_n}\right\}^{-1}$$

$z=t+\mathrm{i}s,\ s>c>0,\ c$ is a certain constant, $\hat{\theta}(z)$ is the measurable solution of the equation

$$\hat{\theta}(z)\frac{1}{n}\mathrm{Tr}\left\{\hat{R}_{m_n}-\hat{\theta}(z)I_{m_n}\right\}^{-1}-\left(1-\frac{m_n}{n}\right)+\frac{\hat{\theta}(z)}{z}=0.$$

THEOREM 26.1. [Gir40, Gir41, Gir50, Gir53, Gir55, Gir57, Gir58] *Suppose $\vec{x}_1,\dots,\vec{x}_n$ is the sample of independent observations of a random vector,*

$$\vec{x}_k=R_{m_n}^{1/2}\vec{\xi}_k+\vec{a},\ \ \mathbf{E}\,\vec{\xi}_k=0,\ \mathbf{E}\,\vec{\xi}_k\vec{\xi}_k^T=I_{m_n},\ \ \vec{\xi}_k^T=\{\xi_{ik},i=1,...,m_n\},\tag{26.1}$$

random variables ξ_{ik} are independent and for a certain $\delta>0$

$$\mathbf{E}\,|\xi_{ik}|^{4+\delta}\le c<\infty,\tag{26.2}$$

$$\lambda_i(R_{m_n}) < c < \infty, \quad i = 1, ..., m_n, \tag{26.3}$$

$$\liminf_{n\to\infty} m_n n^{-1} > 0, \ \limsup_{n\to\infty} m_n n^{-1} < 1, \tag{26.4}$$

$$\lambda_1 (R_{m_n}) > \lambda_k (R_{m_n}) + \tau; \ \tau > 0, \quad k = 2, ..., m; \quad n = 1, 2, ..., \tag{26.5}$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=2}^{m} \frac{\lambda_k^2 (R_{m_n})}{[\lambda_k (R_{m_n}) - \lambda_1 (R_{m_n})]^2} < 1, \tag{26.6}$$

and with probability one

$$\begin{aligned} \liminf_{\varepsilon\downarrow 0} \liminf_{n\to\infty} \Bigg| 1 - \frac{1}{n} \mathrm{Re} \sum_{k=2}^{m} \frac{\lambda_k (R_{m_n})}{\lambda_k (R_{m_n}) - G_{26}^{\max}(A, B, \varepsilon) - \mathrm{i}\varepsilon} \\ + \frac{1}{n} \mathrm{Re} \sum_{k=2}^{m} \frac{G_{26}^{\max} \lambda_k (R_{m_n})}{[\lambda_k (R_{m_n}) - G_{26}^{\max}(A, B, \varepsilon) - \mathrm{i}\varepsilon] [\lambda_k (R_{m_n}) - \lambda_1 (R_{m_n}) - \mathrm{i}\varepsilon]} \Bigg| > 0. \end{aligned} \tag{26.7}$$

Then

$$\lim_{\varepsilon\downarrow 0} \lim_{B\to\infty} \lim_{A\to\infty} p\lim_{n\to\infty} [G_{26}^{\max}(A, B, \varepsilon) - \lambda_1 (R_{m_n})] = 0. \tag{26.8}$$

THEOREM 26.2. *Suppose that conditions (26.1)-(26.4) are fulfilled*

$$\lambda_{m_n} (R_{m_n}) < \lambda_k (R_{m_n}) + \tau; \quad \tau > 0, \quad k = 1, \ldots, m_n - 1; \quad n = 1, 2, ...,$$

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{m_n - 1} \frac{\lambda_k^2 (R_{m_n})}{[\lambda_k (R_{m_n}) - \lambda_{m_n} (R_{m_n})]^2} < 1,$$

and with probability one

$$\begin{aligned} \liminf_{\varepsilon\downarrow 0} \liminf_{n\to\infty} \Bigg| 1 - \frac{1}{n} \mathrm{Re} \sum_{k=1}^{m_n - 1} \frac{\lambda_k (R_{m_n})}{\lambda_k (R_{m_n}) - G_{26}^{\min}(A, B, \varepsilon) - \mathrm{i}\varepsilon} \\ + \frac{1}{n} \mathrm{Re} \sum_{k=1}^{m_n - 1} \frac{G_{26}^{\min}(A, B, \varepsilon) \lambda_k (R_{m_n})}{\left[\lambda_k (R_{m_n}) - G_{26}^{\min}(A, B, \varepsilon) - \mathrm{i}\varepsilon\right] [\lambda_k (R_m) - \lambda_{m_n} (R_{m_n}) - \mathrm{i}\varepsilon]} \Bigg| > 0. \end{aligned}$$

Then

$$\lim_{\varepsilon\downarrow 0} \lim_{B\to\infty} \lim_{A\to\infty} p\lim_{n\to\infty} \left[G_{26}^{\min} (n, A, B, \varepsilon) - \lambda_{m_n} (R_{m_n})\right] = 0.$$

27. G_{27} **- ESTIMATORS OF EIGENVECTORS CORRESPONDING TO EXTREME EIGENVALUES OF THE COVARIANCE MATRIX**

Let $\lambda_1(R_{m_n}) \geq \cdots \geq \lambda_{m_n}(R_{m_n})$ be the eigenvalues and $\vec{\varphi}_1(R_{m_n}), \cdots, \vec{\varphi}_n(R_{m_n})$ be the corresponding orthonormal eigenvectors of the covariance matrix $R_{m_n} = \left(r_{ij}^{(n)}\right)_{i,j=1}^{n}$ and the first nonzero component of every eigenvector is positive. Consider the G-spectral function

$$\nu_n\left(x, R_{m_n}, \vec{b}, \vec{c}\right) = \sum_{k=1}^{m_n} \left[\vec{c}^T \vec{\varphi}_k(R_{m_n})\right]\left[\vec{b}^T \vec{\varphi}_k(R_{m_n})\right] \chi\left[\lambda_k(R_{m_n}) < x\right],$$

where $\vec{b}$ and $\vec{c}$ are arbitrary m_n-dimensional vectors.

The $G_{27}\left\{R_{m_n}, \vec{b}, \vec{c}\right\}$ -consistent estimator for the product of the linear forms $\vec{c}^T \vec{\varphi}_1(R_{m_n}) \vec{b}^T \vec{\varphi}_1(R_{m_n})$, where $\vec{\varphi}_1(R_{m_n})$ is the eigenvector of matrix R_{m_n} corresponding to its maximal eigenvalue, is equal to

$$G_{27}^{(\max)}\left[n, \varepsilon, \delta, A, B, R_{m_n}, \vec{b}, \vec{c}\right] = \frac{1}{2\pi \mathrm{i}} \oint\limits_{\left|u+\mathrm{i}v - G_{26}^{\max}\right| = \delta} G_{25}(A, B, u+\mathrm{i}v) \mathrm{d}(u+\mathrm{i}v),$$

where $\delta > 0$ is a certain small number,

$$\begin{aligned} &G_{25}(A, B, u+\mathrm{i}v) \\ &= \mathrm{i}\int_0^B \left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_0^A \vec{c}^T \mathrm{Im}\left\{\frac{\hat{\theta}(z)}{z}\left[\hat{R}_{m_n} - I_{m_n}\left(\hat{\theta}(z) + \mathrm{i}\varepsilon\right)\right]^{-1}\right\}\vec{b}\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\} \\ &\times \mathrm{e}^{-p(v-\mathrm{i}u)}\chi(v>0)\,\mathrm{d}p, \\ &+ \mathrm{i}\int_0^B \left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^0 \vec{c}^T \mathrm{Im}\left\{\frac{\hat{\theta}(z)}{z}\left[\hat{R}_{m_n} - I_{m_n}\left(\hat{\theta}(z) + \mathrm{i}\varepsilon\right)\right]^{-1}\right\}\vec{b}\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\} \\ &\times \mathrm{e}^{-p(v-\mathrm{i}u)}\chi(v<0)\,\mathrm{d}p. \end{aligned}$$

$z = t + \mathrm{i}s$ and $s > c > 0$, where $c > 0$ is a certain constant.

Similarly we defined the consistent estimator for linear form $\vec{c}^T \vec{\varphi}_{m_n}(R_{m_n}) \vec{b}^T \vec{\varphi}_{m_n}(R_{m_n})$, where $\vec{\varphi}_{m_n}(R_{m_n})$ is the eigenvector of matrix R_{m_n} corresponding to its minimal eigenvalue:

$$G_{27}^{(\min)}\left[n, \varepsilon, \delta, A, B, R_{m_n}, \vec{b}, \vec{c}\right] = \frac{1}{2\pi \mathrm{i}} \oint\limits_{\left|u+\mathrm{i}v - G_{26}^{\min}\right| = \delta} G_{25}(A, B, u+\mathrm{i}v) \mathrm{d}(u+\mathrm{i}v).$$

THEOREM 27.1. *Suppose that conditions (26.1)–(26.7) are fulfilled and*

$$\vec{c}^T \vec{c} + \vec{b}^T \vec{b} \leq c < \infty.$$

Then

$$\lim_{A\to\infty}\lim_{B\to\infty}\underset{n\to\infty}{p\lim}\left[G_{27}^{\max}\left[n,\varepsilon,\delta,A,B,R_{m_n},\vec{b},\vec{c}\right]-\vec{c}^{T}\vec{\varphi}_1\left(R_{m_n}\right)\vec{b}^{T}\vec{\varphi}_1\left(R_{m_n}\right)\right]=0.$$

28. G_{28} CONSISTENT ESTIMATOR OF THE TRACE OF THE RESOLVENT OF THE GRAM MATRIX

The G_{28}-estimator of Stieltjes' transform $\varphi\left(z, AA^T\right) = m_n^{-1}\mathrm{Tr}\left[AA^T - zI_{m_n}\right]^{-1}$ is by definition the following expression:

$$G_{28}\left(z,\Xi\Xi^T\right)=\varphi\left(\hat{\theta}\left(z\right),\Xi\Xi^T\right)\left[1+\gamma\varphi\left(\hat{\theta}\left(z\right),\Xi\Xi^T\right)\right]^{-1}.$$

Here Ξ is an observation of the matrix $A+H$, H is a certain random matrix, $\hat{\theta}(z)$ is the measurable solution of the G_{28}-equation

$$\begin{aligned}&-\hat{\theta}\left(z\right)\left\{1+\frac{1}{n}\mathrm{Tr}\left[\Xi\Xi^T-\hat{\theta}\left(z\right)I_{m_n}\right]^{-1}\right\}^2\\&+\left(1-\frac{m_n}{n}\right)\left\{1+\frac{1}{n}\mathrm{Tr}\left[\Xi\Xi^T-\hat{\theta}\left(z\right)I_{m_n}\right]^{-1}\right\}=-z.\end{aligned}\tag{28.1}$$

Additionally we will use the non-negative solution $\theta(z)$ of the equation

$$\begin{aligned}&-\theta\left(z\right)\left\{1+\frac{1}{n}\mathbf{E}\,\mathrm{Tr}\left[\Xi\Xi^T-\theta\left(z\right)I_{m_n}\right]^{-1}\right\}^2\\&+\left(1-\frac{m_n}{n}\right)\left\{1+\frac{1}{n}\mathbf{E}\,\mathrm{Tr}\left[\Xi\Xi^T-\theta\left(z\right)I_{m_n}\right]^{-1}\right\}=-z.\end{aligned}\tag{28.2}$$

THEOREM 28.1. *If for every n, the entries $\xi_{ij}^{(n)}$, $i=1,\ldots,m_n$; $j=1,\ldots,n$ of random matrix Ξ are independent, $\mathbf{E}\,\xi_{ij}^{(n)}=a_{ij}^{(n)}$, $\mathbf{Var}\,\xi_{ij}^{(n)}=n^{-1}$; for a certain $\delta>0$*

$$\mathbf{E}\left|(\xi_{ij}^{(n)}-a_{ij}^{(n)})\sqrt{n}\right|^{2+\delta}\le c_1<\infty,\quad \max_{i=1,\ldots,m}\sum_{j=1}^{n}a_{ij}^2\le c_2<\infty,$$

$$0<\liminf_{n\to\infty}\frac{m_n}{n}<\limsup_{n\to\infty}\frac{m_n}{n}<1,$$

then with probability one for every $S>0$ and $T>0$

$$\lim_{n\to\infty}\sup_{\substack{0<c\le\mathrm{Im}\,z\le S,\\|\mathrm{Re}\,z|\le T}}\left|G_{28}\left(z\right)-m_n^{-1}\mathrm{Tr}\left\{AA^T-zI_{m_n}\right\}^{-1}\right|=0,$$

where $c>0$ is a certain constant.

Thus, under some conditions

$$\lim_{n\to\infty}\sup_{\substack{0<c\le\mathrm{Im}\,z\le S,\\|\mathrm{Re}\,z|\le T}}\left|G_{28}\left(z\right)-m_n^{-1}\mathrm{Tr}\left\{AA^T-zI_{m_n}\right\}^{-1}\right|=0.$$

We need to know the trace of the resolvent of the covariance matrix for all $s > 0$. Since the function $m_n^{-1}\mathrm{Tr}\left\{R_{m_n} - zI_{m_n}\right\}^{-1}$ is analytic in z, $\mathrm{Im}\, z > 0$ we can use many methods for its analytical continuation. For example, we can use the Fourier transform and consider the following modified G_{28} estimator:

$$G_{28}\left(A, B, u + \mathrm{i}v\right) = \mathrm{i}\int_0^B \left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^{A} \mathrm{Im}\, G_{28}\left(z\right)\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(v-\mathrm{i}u)}\mathrm{d}p,\ v > 0.$$

It is easy to prove that the following assertion is valid: if the conditions of Theorem 28.1 are valid, then with probability one for every $\varepsilon > 0, s > c > 0$

$$\lim_{B\to\infty}\lim_{A\to\infty}\lim_{n\to\infty}\sup_{0<\varepsilon\leq u}\left|G_{28}\left(A, B, u + \mathrm{i}v\right) - m_n^{-1}\mathrm{Tr}\left\{AA^T - \left(u + \mathrm{i}v\right)I_{m_n}\right\}^{-1}\right| = 0.$$

29. G_{29}-CONSISTENT ESTIMATOR OF SINGULAR VALUES OF THE MATRIX

Let $\mu_1 \geq \mu_2 \cdots \geq \mu_{m_n}$ be the eigenvalues and let $h_i, i = 1, ..., m_n;\ h_{1i} > 0$ be the corresponding eigenvectors of the matrix AA^T, $A = (a_{ij})_{i,j=1}^m$.

The $G_{29}^{\max}$ -consistent estimator of maximal eigenvalues $\mu_1\left(AA^T\right)$ of matrix AA^T is equal to a maximal measurable solution x of the equation

$$\mu_1\left(\Xi\Xi^T\right) = x\left\{1 - \mathrm{Re}\left[\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^{A}\mathrm{Im}\, G_{28}\left(z, \Xi X i^T\right)\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(x-\mathrm{i}\varepsilon)}\mathrm{d}p\right]\right\}^2,$$

where Ξ is the observation of matrix $A+H$, H is a random matrix, $z = t+\mathrm{i}s$, $s > c > 0$, $c > 0$ is a certain constant,

$$G_{28}\left(z, \Xi\Xi^T\right) = \frac{1}{m}\mathrm{Tr}\left[\Xi\Xi^T - \hat{\theta}\left(z\right)I_m\right]^{-1}\left[1 + \frac{1}{m}\mathrm{Tr}\left[\Xi\Xi^T - \hat{\theta}\left(z\right)I_m\right]^{-1}\right]^{-1},$$

$\hat{\theta}\left(z\right)$ is the measurable solution of the G_{28} equation

$$\hat{\theta}\left(z\right)\left\{1 + \frac{1}{m}\mathrm{Tr}\left[\Xi\Xi^T - \hat{\theta}\left(z\right)I_{m_n}\right]^{-1}\right\}^2 = z.$$

Similarly we defined the $G_{29}^{\min}$ -consistent estimator for minimal eigenvalues $\mu_m\left(AA^T\right)$ of matrix AA^T which is equal to a minimal measurable solution x of the equation

$$\mu_m\left(\Xi\Xi^T\right) = x\left\{1 - \mathrm{Re}\left[\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-A}^{A}\mathrm{Im}\, G_{28}\left(z\right)\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(x-\mathrm{i}\varepsilon)}\mathrm{d}p\right]\right\}^2.$$

THEOREM 29.1. [Gir55] *Suppose that in addition to the conditions of Theorem 28.1,*

$$m = n,\quad \mathbf{E}\left|(\xi_{ij}^{(n)} - a_{ij}^{(n)})\sqrt{n}\right|^{4+\delta} < c < \infty,\quad \delta > 0$$

and

$$\mu_k\left(AA^T\right) < d < \infty;\ \mu_1\left(AA^T\right) > \mu_k\left(AA^T\right) + \tau;\quad \tau > 0,\quad k = 2, ..., m.$$

$$\limsup_{n\to\infty}\left\{1 - \frac{1}{m}\sum_{k=2}^{m}\frac{\mu_k\left(AA^T\right) + \mu_1\left(AA^T\right)}{\left[\mu_k\left(AA^T\right) - \mu_1\left(AA^T\right)\right]^2}\right\} < 1,$$

and with probability one

$$\liminf_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\left|\left\{1 - \frac{1}{m}\mathrm{Re}\sum_{k=2}^{m}\frac{1}{\mu_k\left(AA^T\right) - \mu_1\left(AA^T\right) - \mathrm{i}\varepsilon}\right\}^2\right.$$
$$+\left\{2 - \frac{1}{m}\mathrm{Re}\sum_{k=2}^{m}\left[\frac{1}{\mu_k\left(AA^T\right) - \mu_1\left(AA^T\right) - \mathrm{i}\varepsilon} + \frac{1}{\mu_k\left(AA^T\right) - G_{29}^{\max} - \mathrm{i}\varepsilon}\right]\right\}$$
$$\left.\times\frac{1}{m}G_{29}^{\max}\mathrm{Re}\sum_{k=2}^{m}\frac{1}{\left[\mu_k\left(AA^T\right) - G_{29}^{\max} - \mathrm{i}\varepsilon\right]\left[\mu_k\left(AA^T\right) - \mu_1\left(AA^T\right) - \mathrm{i}\varepsilon\right]}\right| > 0.$$

Then

$$\lim_{\varepsilon\downarrow 0}\lim_{B\to\infty}\lim_{C\to\infty}\operatorname*{plim}_{n\to\infty}\left[G_{29}^{\max}(B, C, \varepsilon) - \mu_1\left(AA^T\right)\right] = 0.$$

30. G_{30}-CONSISTENT ESTIMATOR OF EIGENVECTORS CORRESPONDING TO EXTREME SINGULAR VALUES OF THE MATRIX

Let $\lambda_1\left(AA^T\right) \geq \cdots \geq \lambda_m\left(AA^T\right)$ be eigenvalues and $\vec{\varphi}_1\left(AA^T\right), \cdots, \vec{\varphi}_m\left(AA^T\right)$ be corresponding orthonormal eigenvectors of the matrix AA^T, $A = \left(a_{ij}^{(n)}\right)_{i,j=1,\ldots,m_n}$ and the first nonzero component of every eigenvector is positive. Consider G-spectral function

$$\nu_n\left(x, AA^T, \vec{b}, \vec{c}\right) = \sum_{k=1}^{m_n}\left[\vec{c}^T\vec{\varphi}_k\left(AA^T\right)\right]\left[\vec{b}^T\vec{\varphi}_k\left(AA^T\right)\right]\chi\left[\lambda_k\left(AA^T\right) < x\right],$$

where $\vec{b}$ and $\vec{c}$ are arbitrary m_n-dimensional vectors.

Suppose we have one observation $\Xi = \left(\xi_{ij}^{(n)}\right)_{i,j=1,\ldots,m_n}$ of matrix $A + H$, where H is a random matrix.

The $G_{30}\left\{\Xi, \vec{b}, \vec{c}\right\}$ -consistent estimator for the product of the linear forms $\vec{c}^T\vec{\varphi}_1\left(AA^T\right)\vec{b}^T\vec{\varphi}_1\left(AA^T\right)$, where $\vec{\varphi}_1\left(AA^T\right)$ is the eigenvector of matrix AA^T corresponding to its maximal eigenvalue is equal to

$$G_{30}^{(\max)}\left[n, \varepsilon, \delta, \Xi, \vec{b}, \vec{c}\right] = \frac{1}{2\pi\mathrm{i}}\oint_{\left|u+\mathrm{i}v - G_{29}^{\max}\right| = \delta} G\left(C, B, u + \mathrm{i}v\right)\mathrm{d}\left(u + \mathrm{i}v\right),$$

where $\delta > 0$ is a certain small number,

$$
\begin{aligned}
G(C,B,u+\mathrm{i}v) &= \mathrm{i}\int_0^B \left\{ \frac{\mathrm{e}^{|sp|}}{\pi} \int_0^C \vec{c}^T \mathrm{Im} \frac{1}{1+n^{-1}\mathrm{Tr}\left\{\Xi\Xi^T - I\left[\hat{\theta}(z)+\mathrm{i}\varepsilon\right]\right\}^{-1}} \right. \\
&\quad \times \left. \left\{\Xi\Xi^T - I\left[\hat{\theta}(z)+\mathrm{i}\varepsilon\right]\right\}^{-1} \vec{b}\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t \right\} \mathrm{e}^{-p(v-\mathrm{i}u)}\chi(v>0)\,\mathrm{d}p \\
&\quad + \mathrm{i}\int_0^B \left\{ \frac{\mathrm{e}^{|sp|}}{\pi} \int_{-A}^0 \vec{c}^T \mathrm{Im} \left\{ \frac{1}{1+n^{-1}\mathrm{Tr}\left[\Xi - I\left(\hat{\theta}(z)+\mathrm{i}\varepsilon\right)\right]^{-1}} \right. \right. \\
&\quad \times \left. \left. \left[\Xi - I\left(\hat{\theta}(z)+\mathrm{i}\varepsilon\right)\right]^{-1} \right\} \vec{b}\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t \right\} \mathrm{e}^{-p(v-\mathrm{i}u)}\chi(v<0)\,\mathrm{d}p,
\end{aligned}
$$

Similarly, we defined the consistent estimator for the linear form $\vec{c}^T\vec{\varphi}_{m_n}\left(AA^T\right)\vec{b}^T\vec{\varphi}_{m_n}\left(AA^T\right)$, where $\vec{\varphi}_{m_n}\left(AA^T\right)$ is the eigenvector of matrix AA^T corresponding to its minimal eigenvalue:

$$
G_{30}^{(\min)}\left[n,\varepsilon,\delta,C,B,\Xi,\vec{b},\vec{c}\right] = \frac{1}{2\pi\mathrm{i}} \oint\limits_{\left|u+\mathrm{i}v-G_{29}^{\min}\right|=\delta} G(C,B,u+\mathrm{i}v)\,\mathrm{d}(u+\mathrm{i}v).
$$

THEOREM 30.1. [Gir55] *Suppose that the conditions of Theorem 29.1 are fulfilled and*

$$
\vec{c}^T\vec{c} + \vec{b}^T\vec{b} \le c < \infty.
$$

Then

$$
\lim_{C\to\infty}\lim_{B\to\infty}\lim_{n\to\infty}\left\{G_{30}^{\max}\left[n,\varepsilon,\delta,C,B,\Xi,\vec{b},\vec{c}\right] - \vec{c}^T\vec{\varphi}_1\left(AA^T\right)\vec{b}^T\vec{\varphi}_1\left(AA^T\right)\right\} = 0.
$$

31. G_{31}-ESTIMATOR OF THE RESOLVENT OF A SYMMETRIC MATRIX

In this section we explain the main ideas of estimation of eigenvalues and eigenvectors of matrices. The G_{31}-estimator of Stieltjes' transform $\varphi(z,A) = n^{-1}\mathrm{Tr}\left[A - zI_n\right]^{-1}$ where A is a symmetric matrix, $z = t + \mathrm{i}s,\ s > 0$ is by definition the following expression:

$$
G_{31}(z,\Xi) = \varphi\left(\hat{\theta}(z),\Xi\right) = n^{-1}\mathrm{Tr}\left[\Xi - \hat{\theta}(z)I_n\right]^{-1},
$$

where Ξ is an observation of matrix $A+H,\ H$ is a random matrix, $\hat{\theta}(z)$ is the measurable solution of the G_{31} equation

$$
\hat{\theta}(z) + n^{-1}\mathrm{Tr}\left[\Xi - \hat{\theta}(z)I_n\right]^{-1} = z, z = t + is, s > 0. \tag{31.1}
$$

31.1. Canonical equation

THEOREM 31.1. [Gir55] *If for every n, the random entries $\xi_{ij}^{(n)},\ i \geq j;\ i,j=1,\dots,n$ of a symmetric matrix $\Xi = \left(\xi_{ij}^{(n)}\right)_{i=1,\dots,n}^{j=1,\dots,n}$ are independent, $\mathbf{E}\,\xi_{ij}^{(n)} = a_{ij}^{(n)};\ \ \mathbf{Var}\,\xi_{ij}^{(n)} = n^{-1}$,*

$$\sup_n \max_{k=1,\dots,m} \sum_{j=1}^{n} a_{kj}^2 < \infty,$$

the modified Lindeberg's condition is fulfilled: for a certain $\tau > 0$

$$\lim_{n\to\infty} n^{-1} \sum_{i,j=1}^{n} \mathbf{E} \left|\xi_{ij}^{(n)}\right|^2 \chi\left(\left|\xi_{ij}^{(n)}\right| > \tau\right) = 0$$

or

$$\sup_n \max_{i,j=1,\dots,n} \mathbf{E} \left|\left(\xi_{ij}^{(n)} - \mathbf{E}\,\xi_{ij}^{(n)}\right)\sqrt{n}\right|^{2+\delta} < \infty \tag{31.2}$$

and $|\alpha_i(A)| < c < \infty;\ A = \left(a_{ij}^{(n)}\right)_{i=1,\dots,n}^{j=1,\dots,n}$, where $\alpha_1(A) \leq \dots \leq \alpha_n(A)$ are the eigenvalues of the matrix A, then with probability 1

$$\lim_{n\to\infty} \sup_x |\mu_n(x) - F_n(x)| = 0, \tag{31.3}$$

where $\mu_n(x) = n^{-1}\sum_{k=1}^{n} \chi(\lambda_k(\Xi) < x)$ and $F_n(x)$ is the distribution function:

$$F_n(x) = \int_{-\infty}^{x} p_n(y)\mathrm{d}y. \tag{31.4}$$

Here, the density $p(x)$ is the first component of the real vector-solution $\{p(x),\ g(x)\}$ of the system of canonical equations [Gir84]

$$\begin{aligned} 1 &= n^{-1}\sum_{k=1}^{n} [(\alpha_k^{(n)} - x - g(x))^2 + \pi^2 p^2(x)]^{-1}, \\ g(x) &= n^{-1}\sum_{k=1}^{n} (\alpha_k^{(n)} - x - g(x))[(\alpha_k^{(n)} - x - g(x))^2 + \pi^2 p^2(x)]^{-1}. \end{aligned} \tag{31.5}$$

There exists a unique solution of a system of equations (31.5) in the class of functions

$$B = \left\{p(x), g(x)\ :\ \ p(x) \in G,\ p(x) > 0;\ \ x > 0;\quad \int_G p(x)\mathrm{d}x = 1\right\}$$

for every x for which $p(x) > 0$. Stieltjes' transform

$$b(z) = \int (x-z)^{-1} p(x)\,\mathrm{d}x,\ \ z = t + \mathrm{i}s, s > 0$$

of function $p(x)$ satisfies the equation (see [Pas1])

$$b(z) = n^{-1} \sum_{k=1}^{n} [\alpha_k - z - b\,(z)]^{-1}, \tag{31.6}$$

which has the unique solution $b(z)$ in the class of analytic functions:
$B_1 = \{b\,(z) : \text{Im}\,[b(z)] > 0, \text{Im}\, z > 0\}$ *and can be obtained by the method of successive approximations.*

Corollary 31.1. [Pas1] *If in addition to the conditions of Theorem 31.1 $\alpha_k^{(n)} = 0$, $k = 1, ..., n$, then the component $p(x)$ of the solution of the system of equations (31.5) is equal to*

$$p(x) = (2\pi)^{-1}(4 - x^2)^{1/2}, \ \ |x| \le 2$$

and this is called the semicircle Wigner Law.

31.2. Equations for boundary points of spectral density

Denote

$$\gamma_1 = \inf_x \{x : \ \ p\,(x) > 0\}\,, \ \ \gamma_2 = \sup_x \{x : \ \ p\,(x) > 0\}\,.$$

Lemma 31.1. [Gir84] *Assume that the condition*

$$|\alpha_k| \le c < \infty, \ \ k = 1, \ldots, n$$

holds. Then

$$-\infty < c_1 < \gamma_1 < \gamma_2 < c_2 < \infty,$$

where

$$\gamma_i = \nu_i - n^{-1} \sum_{k=1}^{n} \left(\alpha_k^{(n)} - \nu_i\right)^{-1}, \ \ i = 1, 2,$$

$$v_1 = \min y_i, v_2 = \max y_i$$

and y_i are the real solutions of L_1 equation

$$n^{-1} \sum_{k=1}^{n} \left(\alpha_k^{(n)} - y\right)^{-2} = 1.$$

31.3. Inequality for imaginary parts of resolvent of symmetric random matrix

Theorem 31.2. *If the conditions of Theorem 31.1 are fulfilled, then for all $T > 0$, $s > c$, $\varepsilon > 0$*

$$\limsup_{n\to\infty} \sup_{|t|\le T,\ 0<\varepsilon<s\le S} \left| n^{-1} \text{Im}\, \mathbf{E}\ \text{Tr}\, (\Xi_n - I_n z)^{-1} \right| \le c_1.$$

Proof. Using the equation

$$b(z) = n^{-1} \sum_{k=1}^{n} [\alpha_k - z - b(z)]^{-1}$$

and Theorem 31.1 we arrive to the assertion of Theorem 31.2

31.4. Existence and boundedness of the solutions of the main equation

The G_{31}-estimator of Stieltjes' transform $\varphi(z, A) = n^{-1}\mathrm{Tr}\,[A - zI_n]^{-1}$ is by definition the following expression: $G_{31}(z, \Xi) = \varphi\left(\hat{\theta}(z), \Xi\right) = n^{-1}\mathrm{Tr}\left[\Xi - \hat{\theta}(z) I_n\right]^{-1}$, where $\hat{\theta}(z)$ is the measurable solution of a main G_{31}-equation

$$\hat{\theta}(z) + n^{-1}\mathrm{Tr}\left[\Xi - \hat{\theta}(z) I_n\right]^{-1} = z. \tag{31.7}$$

Additionally, we will use the non-negative solution $\theta(z)$ of the equation

$$\theta(z) + n^{-1}\mathbf{E}\ \mathrm{Tr}\,[\Xi - \theta(z) I_n]^{-1} = z. \tag{31.8}$$

THEOREM 31.3. *Let $s > 1$. Then for large n with probability one, there exists the solution $\hat{\theta}(z)$ of the equation (31.7) and the solution $\theta(z)$ of the equation (31.8).*

Proof. Denote $\hat{\theta}(z) = \hat{\theta}_1(z) + \mathrm{i}\hat{\theta}_2(z)$. Then equation (31.7) has the form

$$\hat{\theta}_1(z) + n^{-1}\mathrm{Re}\,\mathrm{Tr}\left[\Xi - \hat{\theta}(z) I_n\right]^{-1} = t,$$

$$\hat{\theta}_2(z) + n^{-1}\mathrm{Im}\,\mathrm{Tr}\left[\Xi - \hat{\theta}(z) I_n\right]^{-1} = s.$$

Let $\hat{\theta}_2(z) > c > 0$ be a fixed number, where $c > 0$ is a certain number. Then there always exists the solution of (31.7). For the second equation we have

$$\hat{\theta}_2(z) = s - n^{-1}\mathrm{Im}\,\mathrm{Tr}\left[\Xi - \hat{\theta}(z) I_n\right]^{-1}. \tag{31.9}$$

But from Theorem 31.2 it follows that for large n, there exists such $\varepsilon > 0$ that

$$\mathbf{P}\left\{0 \le \sup_{0 < c_1 \le \hat{\theta}_2(z)} \mathrm{Im}\,\frac{1}{n}\mathrm{Tr}\left\{\Xi - \hat{\theta}(z) I_n\right\}^{-1} \le 1 + \varepsilon\right\} = 1.$$

Therefore, choosing $\mathrm{Im}\,z > 1$, we prove that for large n, there exists the solution $\hat{\theta}_2(z) > \delta > 0$ of equation (31.9) with probability one. Therefore, for large n, there exist solutions $\hat{\theta}(z)$, $\hat{\theta}_2(z) > \delta > 0$ of equation (31.7) with probability one.

Analogously we prove that for large n there exists a solution of (31.8).

31.5. Ordered solutions of the main equations

In the general case for large n with probability one there are many solutions of equations (31.7) and (31.8). We denote them $\hat{\theta}_1(z), \hat{\theta}_2(z), \ldots$ and $\theta_1(z), \theta_2(z), \ldots$ respectively, where

$$\left|\hat{\theta}_1(z)\right| \ge \left|\hat{\theta}_2(z)\right| \ge \left|\hat{\theta}_3(z)\right| \ge \cdots;\ |\theta_1(z)| \ge |\theta_2(z)| \ge |\theta_3(z)| \ge \cdots.$$

Theorem 31.4. *If the conditions of Theorem 31.1 are fulfilled, then there exists constant $c > 0$ such that for every $s > c$ and t*

$$\mathbf{P}\left\{\lim_{n\to\infty}\left|\hat{\theta}_1(z) - \theta_1(z)\right| = 0\right\} = 1.$$

Proof. From Theorem 31.3 we obtain

$$\mathbf{P}\left\{\lim_{n\to\infty}\sup_{|t|\le T;\ 0<c_1\le s<S}\left|\frac{1}{n}\mathrm{Tr}\left\{\Xi - zI_n\right\}^{-1} - \frac{1}{n}\mathbf{E}\,\mathrm{Tr}\left\{\Xi - zI_n\right\}^{-1}\right| = 0\right\} = 1.$$

Therefore, we can change the equation

$$\hat{\theta}(z) + n^{-1}\mathrm{Tr}\left[\Xi - \hat{\theta}(z)\,I_n\right]^{-1} = z$$

to

$$\theta(z) + n^{-1}\mathbf{E}\,\mathrm{Tr}\left[\Xi - \theta(z)\,I_n\right]^{-1} = z + \varepsilon_n,$$

where $\varepsilon_n \to 0$ with probability one. Then, denoting

$$f_n(\theta(z)) = n^{-1}\mathbf{E}\ \mathrm{Tr}\left[\Xi - \theta(z)\,I_n\right]^{-1} - z,$$

we obtain that $\theta_1(z) + f_n(\theta_1(z)) = \hat{\theta}_1(z) + f_n\left(\hat{\theta}_1(z)\right) + \varepsilon_n$ and the function $f_n(z)$, $\mathrm{Im}\, z > c > 0$ and its every convergent limit are analytic functions on the domain $\{z : \mathrm{Im}\, z > c > 0\}$.

Therefore,

$$\mathbf{P}\left\{\lim_{n\to\infty}\left|\hat{\theta}_1(z) - \theta_1(z)\right| = 0\right\} = 1$$

and Theorem 31.4 is proved.

31.6. Consistency of estimator G_{31}

Theorem 31.5. *If the conditions of Theorem 31.1 are fulfilled, then with probability one, for every $S > 0$ and $T > 0$*

$$\lim_{n\to\infty}\sup_{1<\mathrm{Im}\, z\le S,\ |\mathrm{Re}\, z|\le T}\left|G_{31}(z) - n^{-1}\mathrm{Tr}\left\{A - zI_n\right\}^{-1}\right| = 0.$$

Proof. Using Theorem 31.4 we have

$$\mathbf{P}\left\{\lim_{n\to\infty}\left|\hat{\theta}_1(z) - \theta_1(z)\right| = 0\right\} = 1,$$

where $\hat{\theta}_1(z)$ and $\theta_1(z)$ are solutions of the equations $\hat{\theta}(z) + n^{-1}\mathrm{Im}\,\mathrm{Tr}\left[\Xi - \hat{\theta}(z)\,I_n\right]^{-1} = z$ and $\theta(z) + n^{-1}\mathbf{E}\,\mathrm{Tr}\left[\Xi - \theta(z)\,I_n\right]^{-1} = z$ respectively. Hence, using Theorem 31.1 we obtain with probability one

$$\begin{aligned}G_{31}(z,\Xi) &= n^{-1}\mathrm{Im\,Tr}\left[\Xi-\hat{\theta}(z)I_n\right]^{-1}\\ &= n^{-1}\mathrm{Im\,Tr}\left[\Xi-\theta(z)I_n\right]^{-1}+\varepsilon_n\\ &= \frac{1}{n}\sum_{k=1}^{n}\left\{\alpha_k-\theta(z)-n^{-1}\mathbf{E}\,\mathrm{Tr}\left[\Xi-\theta(z)I_n\right]^{-1}\right\}^{-1}+\varepsilon_n\\ &= \frac{1}{n}\mathrm{Tr}\left[A-zI_n\right]^{-1}+\tilde{\varepsilon}_n,\end{aligned}$$

where $\mathbf{P}\left\{\lim_{n\to\infty}\left[|\tilde{\varepsilon}_n|+|\varepsilon_n|\right]=0\right\}=1$. Theorem 31.5 is proved.

31.7. Analytical continuation of estimator G_{31}

Thus, under some conditions

$$\lim_{n\to\infty}\sup_{1<\mathrm{Im}\,z\le S,\ |\mathrm{Re}\,z|\le T}\left|G_{31}(z)-n^{-1}\mathrm{Tr}\left\{A-zI_n\right\}^{-1}\right|=0.$$

One needs to know the trace of the resolvent of the covariance matrix for all $s>0$. Since the function $n^{-1}\mathrm{Tr}\left\{A_n-zI_n\right\}^{-1}$ is analytical in $z:\ \mathrm{Im}\,z>0$, we can use methods of its analytical continuation. For example, we can use the Fourier transform and consider the following modified G_{31} estimator:

$$G_{31}(C,B,u+\mathrm{i}v)=\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-C}^{C}\mathrm{Im}\,G_{31}(z)\,\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(v-\mathrm{i}u)}\mathrm{d}p,\ \ v>0.$$

It is easy to prove that the following assertion is valid: if the conditions of Theorem 31.1 are valid then with probability one for every $\varepsilon>0, s>c>0$

$$\lim_{B\to\infty}\lim_{C\to\infty}\lim_{n\to\infty}\sup_{\nu,\,0<\varepsilon\le u}\left|G_{31}(C,B,u+\mathrm{i}v)-n^{-1}\mathrm{Tr}\left\{A-(u+\mathrm{i}v)I_n\right\}^{-1}\right|=0.$$

32. G_{32}-ESTIMATOR OF EIGENVALUES OF A SYMMETRIC MATRIX

Let $\lambda_1\le\cdots\le\lambda_n$ be eigenvalues of the symmetric random matrix $\Xi=\left(\xi_{ij}^{(n)}\right)_{i,j=1}^{n}$.

THEOREM 32.1 [Gir55] *Assume that the entries $\xi_{ij}^{(n)},\ i\ge j,\ i,j=1,\dots,n$ are independent for every n*

$$\mathbf{E}\,\xi_{ij}^{(n)}=a_{ij}^{(n)},\ \mathbf{E}\left[\xi_{ij}^{(n)}-a_{ij}^{(n)}\right]^2=n^{-1},$$

for a certain $\beta>0$

$$\sup_n\sup_{i,j=1,\dots,n}\mathbf{E}\left|\left(\xi_{ij}^{(n)}-a_{ij}^{(n)}\right)n^{1/2}\right|^{8+\beta}<\infty,$$

$$\left|\alpha_k^{(n)}\right|\le c<\infty,\quad k=1,\dots,n,$$

where $\alpha_1^{(n)} \leq \cdots \leq \alpha_n^{(n)}$ *are eigenvalues of the matrix* $A = \left(a_{ij}^{(n)}\right)_{i,j=1}^n$,

$$\max_{k=1,\ldots,n} \sum_{j=1}^n a_{kj}^2 \leq c < \infty.$$

Then with probability 1
1).

$$\lim_{n\to\infty} [\lambda_1 - \gamma_1] = 0, \quad \lim_{n\to\infty} [\lambda_n - \gamma_2] = 0,$$

where

$$\gamma_i = v_i - \frac{1}{n} \sum_{k=1}^n \left(\alpha_k^{(n)} - v_i\right)^{-1}; \ \ i = 1, 2, \tag{32.1}$$

$v_1 = \min y_i, v_2 = \max y_i$ *and* y_i *are the real solutions of the* L_1 *equation*

$$\frac{1}{n} \sum_{k=1}^n \left(\alpha_k^{(n)} - y_i\right)^{-2} = 1.$$

2). For all k *such that* $0 < c_1 \leq \frac{k}{n} \leq c_2 < 1$ *with probability 1*

$$\lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \left|\lambda_k - \inf\left\{x : \frac{k}{n} - \varepsilon \leq F(x)\right\}\right| \left|\lambda_k - \sup\left\{x : F(x) \leq \frac{k}{n} + \varepsilon\right\}\right| = 0. \tag{32.2}$$

(see formula (31.4) for function $F(x)$ *)*
3). If the limits

$$\lim_{n\to\infty} kn^{-1} = y, \ \ \lim_{n\to\infty} F_n(x) = F(x)$$

exist for a certain k *and if* $F(x)$ *is an increasing function in some neighborhood of the point* y*, then with probability one* $\lim_{n\to\infty} \left[\lambda_k - F^{(-1)}(y)\right] = 0$*, where* $F^{(-1)}(y)$ *is the inverse function.*

The $G_{32}^{\max}$ *-consistent estimator for maximal eigenvalues* $\lambda_1(A)$ *of matrix* A *is equal to a maximal measurable solution* x *of the equation*

$$\lambda_1(\Xi) = x - \mathrm{Re}\left[\mathrm{i}\int_0^B \left\{\frac{\mathrm{e}^{|sp|}}{\pi} \int_{-C}^{C} \mathrm{Im}\, G_{32}(z)\, \mathrm{e}^{-\mathrm{i}tp} \mathrm{d}t\right\} \mathrm{e}^{-p(x - \mathrm{i}\varepsilon)} \mathrm{d}p\right].$$

Here, Ξ *is the observation of matrix* $A + H$*,* H *is a random matrix,*

$$G_{32}(z) = n^{-1} \mathrm{Tr}\left[\Xi - \hat{\theta}(z) I_n\right]^{-1},$$

$\hat{\theta}(z)$ *is the measurable solution of the* G_{32} *equation*

$$\hat{\theta}(z) + n^{-1} \mathrm{Tr}\left[\Xi - \hat{\theta}(z) I_n\right]^{-1} = z.$$

Similarly, we defined the $G_{32}^{\min}$ -consistent estimator for the minimal eigenvalue $\lambda_n(A)$ of a matrix A, which is equal to a minimal measurable solution x of the equation

$$\lambda_n(\Xi) = x - \operatorname{Re}\left[\mathrm{i}\int_0^B \left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-C}^{C} \operatorname{Im} G_{32}(z)\,\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(x-\mathrm{i}\varepsilon)}\mathrm{d}p\right].$$

THEOREM 32.2. [Gir55] *Suppose that the conditions of Theorem 32.1 are fulfilled and*

$$|\lambda_k(A)| < d < \infty;\ \lambda_1(A) > \lambda_k(A) + \tau;\ \tau > 0, k = 2, \ldots, n,$$

$$\liminf_{n\to\infty}\left\{1 - \frac{1}{n}\sum_{k=2}^{n}\frac{1}{[\lambda_k(A) - \lambda_1(A)]^2}\right\} > 0,$$

and with probability one

$$\liminf_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\left|1 - \frac{1}{n}\operatorname{Re}\sum_{k=2}^{n}\frac{1}{[\lambda_k(A) - \lambda_1(A) + \mathrm{i}\varepsilon][\lambda_k(A) - G_{32}(z) + \mathrm{i}\varepsilon]}\right| > 0.$$

Then

$$\lim_{\varepsilon\downarrow 0}\lim_{B\to\infty}\lim_{C\to\infty}\lim_{n\to\infty}[G_{32}^{\max}(n, C, B, \varepsilon) - \lambda_1(A)] = 0.$$

33. G_{33}-ESTIMATOR OF THE EIGENVECTOR WHICH CORRESPONDS TO EXTREME EIGENVALUES OF THE SYMMETRIC MATRIX

Let $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ be the eigenvalues and $\vec{\varphi}_1(A), \cdots, \vec{\varphi}_n(A)$ be the corresponding orthonormal eigenvectors of the symmetric matrix $A = \left(a_{ij}^{(n)}\right)_{i,j=1}^{n}$ and suppose that the first nonzero component of every eigenvector is positive. Consider the G-spectral function

$$\nu_n\left(x, A, \vec{b}, \vec{c}\right) = \sum_{k=1}^{n}\left[\vec{c}^T\vec{\varphi}_k(A)\right]\left[\vec{b}^T\vec{\varphi}_k(A)\right]\chi[\lambda_k(A) < x],$$

where $\vec{b}$ and $\vec{c}$ are arbitrary n-dimensional vectors.

Suppose we have one observation $\Xi = \left(\xi_{ij}^{(n)}\right)_{i,j=1}^{n}$ of matrix $A = \left(a_{ij}^{(n)}\right)_{i,j=1}^{n} + \left(\eta_{ij}^{(n)}\right)_{i,j=1}^{n}$.

The $G_{33}\left\{\Xi, \vec{b}, \vec{c}\right\}$ -consistent estimator for the product of the linear forms $\vec{c}^T\vec{\varphi}_1(A)\,\vec{b}^T\vec{\varphi}_1(A)$, where $\vec{\varphi}_1(A)$ is the eigenvector of matrix A corresponding to its maximal eigenvalue, is equal to

$$G_{33}^{(\max)}\left[n, \varepsilon, \delta, \Xi, \vec{b}, \vec{c}\right] = \frac{1}{2\pi\mathrm{i}}\oint\limits_{\left|u+\mathrm{i}v - G_{32}^{\max}\right| = \delta} G(C, B, u+\mathrm{i}v)\mathrm{d}(u+\mathrm{i}v),$$

where $\delta > 0$ is a certain small number,

$$
\begin{aligned}
&G\left(C,B,u+\mathrm{i}v\right)\\
&=\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_0^C\vec{c}^T\mathrm{Im}\left\{\Xi-I\left[\hat{\theta}\left(z\right)+\mathrm{i}\varepsilon\right]\right\}^{-1}\vec{b}\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(v-\mathrm{i}u)}\chi\left(v>0\right)\mathrm{d}p\\
&+\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-C}^0\vec{c}^T\mathrm{Im}\left\{\Xi-I\left[\hat{\theta}\left(z\right)+\mathrm{i}\varepsilon\right]\right\}^{-1}\vec{b}\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(v-iu)}\chi\left(v<0\right)\mathrm{d}p,
\end{aligned}
$$

$G_{32}^{\max}$ is a maximal measurable solution x of the equation

$$
\lambda_1\left(\Xi\right)=x-\mathrm{Re}\left[\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-C}^C\mathrm{Im}\,G_{31}\left(z\right)\mathrm{e}^{-\mathrm{i}tp}dt\right\}\mathrm{e}^{-p(x-\mathrm{i}\varepsilon)}\mathrm{d}p\right],
$$

$$
G_{31}\left(z\right)=n^{-1}\mathrm{Tr}\left[\Xi-\hat{\theta}\left(z\right)I_n\right]^{-1},
$$

and $\hat{\theta}\left(z\right)$ is the measurable solution of the G_{31}-equation

$$
\hat{\theta}\left(z\right)+n^{-1}\mathrm{Tr}\left[\Xi-\hat{\theta}\left(z\right)I_n\right]^{-1}=z.
$$

Similarly, we defined the consistent estimator for linear form $\vec{c}^T\vec{\varphi}_n\left(A\right)\vec{b}^T\vec{\varphi}_n\left(A\right)$, where $\vec{\varphi}_n\left(A\right)$ is the eigenvector of matrix A corresponding to its minimal eigenvalue:

$$
G_{33}^{(\min)}\left[n,\varepsilon,\delta,C,B,\Xi,\vec{b},\vec{c}\right]=\frac{1}{2\pi\mathrm{i}}\oint\limits_{\left|u+\mathrm{i}v-G_{32}^{\min}\right|=\delta}G\left(C,B,u+\mathrm{i}v\right)\,\mathrm{d}\left(u+\mathrm{i}v\right).
$$

Here $G_{32}^{\min}$ is the consistent estimator for minimal eigenvalues $\lambda_n\left(A\right)$ of matrix A which is equal to a minimal measurable solution x of the equation

$$
\lambda_n\left(\Xi\right)=x-\mathrm{Re}\left[\mathrm{i}\int_0^B\left\{\frac{\mathrm{e}^{|sp|}}{\pi}\int_{-C}^C\mathrm{Im}\,G_{31}\left(z\right)\mathrm{e}^{-\mathrm{i}tp}\mathrm{d}t\right\}\mathrm{e}^{-p(x-\mathrm{i}\varepsilon)}\mathrm{d}p\right].
$$

THEOREM 33.1. *Suppose that the conditions of Theorem 32.1 are fulfilled and*

$$
\vec{c}^T\vec{c}+\vec{b}^T\vec{b}\leq c<\infty,
$$

$$
\left|\lambda_k\left(A\right)\right|<d<\infty;\ \lambda_1\left(A\right)>\lambda_k\left(A\right)+\tau;\quad\tau>0,k=2,...,n,\quad\lambda_2(A)<\alpha_2,
$$

$\lambda_2\left(A\right)<\alpha_2,$

$$
\liminf_{n\to\infty}\left\{1-\frac{1}{n}\sum_{k=2}^n\frac{1}{\left[\lambda_k\left(A\right)-\lambda_1\left(A\right)\right]^2}\right\}>0,
$$

and with probability one

$$\liminf_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\left|1-\frac{1}{n}\mathrm{Re}\sum_{k=2}^{n}\frac{1}{\left[\lambda_k(A)-\lambda_1(A)+\mathrm{i}\varepsilon\right]\left[\lambda_k(A)-G_{32}^{\max}(z)+\mathrm{i}\varepsilon\right]}\right|>0.$$

Then

$$\lim_{C\to\infty}\lim_{B\to\infty}\lim_{n\to\infty}\left[G_{33}^{\max}\left[n,\varepsilon,\delta,C,B,\Xi,\vec{b},\vec{c}\right]-\vec{c}^{T}\vec{\varphi}_1(A)\,\vec{b}^{T}\vec{\varphi}_1(A)\right]=0.$$

The proof of this Theorem is based on the use of the asymptotic expression for the traces of the resolvents of the matrix Ξ.

34. G_{34}-ESTIMATOR OF THE V-TRANSFORM

Here we consider the most difficult problem of estimation of eigenvalues and eigenvectors of matrices, i.e. the case when the matrices are nonsymmetric.

In this section we assume that the expectation of the entries of the random matrices may not equal zero. Let us consider matrices $A_n+\Xi_n$, where $A_n=(a_{ij})_{i,j=1}^{n}$ is a nonrandom complex matrix and Ξ_n is a random matrix.

THEOREM 34.1. [Gir73, Gir84] (V-Law). *Let $\Xi=(\xi_{ij}^{n})_{i,j=1,\dots,n}$ be random complex matrices whose entries $\xi_{ij}^{(n)}$ $i\geq j$ are independent for every n, $\mathbf{E}\,\xi_{ij}^{(n)}=0$; $\mathbf{E}\,|\xi_{ij}^{(n)}|^2=n^{-1}$, $\mathbf{E}\,\xi_{ij}^{(n)}\xi_{ji}^{(n)}=\rho n^{-1}$, $i\neq j$ and for some $\delta>0$*

$$\mathbf{E}\,|\xi_{ij}^{(n)}\sqrt{n}|^{4+\delta}\leq c<\infty,$$

and suppose that there exist densities $p_{ij}^{(n)}(x,y,u,v)$ of the random entries $\sqrt{n}\mathrm{Re}\,\xi_{ij}^{(n)}$,
$\sqrt{n}\mathrm{Im}\xi_{ij}^{(n)}$,
$\sqrt{n}\mathrm{Re}\xi_{ji}^{(n)}$, $\sqrt{n}\mathrm{Im}\xi_{ji}^{(n)}$, $i>j$ satisfying the condition: for some $\beta>1$

$$\sup_{n}\max_{\substack{k,l=1,\dots,n\\k\neq l}}\iint\left[\int\left[\int p_{kl}^{(n)}(x,y,u,v)\mathrm{d}y\right]^{\beta}\mathrm{d}x\right]^{1/\beta}\mathrm{d}u\,\mathrm{d}v<\infty,$$

or

$$\sup_{n}\max_{\substack{k,l=1,\dots,n\\k\neq l}}\iint\left[\int\left[\int p_{kl}^{(n)}(x,y,u,v)\mathrm{d}x\right]^{\beta}\mathrm{d}y\right]^{1/\beta}\mathrm{d}u\,\mathrm{d}v<\infty,$$

and that there exist the densities $p_{ii}^{(n)}(x)$ of the random entries $\sqrt{n}\Re\xi_{ii}^{(n)}$, or the densities $q_{ii}^{(n)}(x)$ of the random entries $\sqrt{n}\Im\xi_{ii}^{(n)}$, satisfying the condition: for some $\beta_1>1$

$$\sup_{n}\max_{k=1,\dots,n}\int\left[p_{kk}^{(n)}(x)\right]^{\beta_1}\mathrm{d}x<\infty,\quad\text{or}\quad\sup_{n}\max_{k=1,\dots,n}\int\left[q_{kk}^{(n)}(x)\right]^{\beta_1}\mathrm{d}x<\infty,$$

$$\sup_{n}\max_{i,j=1,\dots,n}\sum_{i=1}^{n}\left[|a_{ij}|+|a_{ji}|\right]<\infty,$$

then

$$\underset{n\to\infty}{p\lim}|\mu_n(x,y)-F_n(x,y)|=0.$$

Here

$$\mu_n(x,y)=n^{-1}\sum_{k=1}^{n}\chi(\operatorname{Re}\lambda_k<x,\ \operatorname{Im}\lambda_k(y)),$$

are eigenvalues of matrix $A_n+\Xi_n$, *the* $\mathbf{V}$*-density* $p_n(x,y)=\frac{\partial^2}{\partial x\partial y}F_n(x,y)$ *is equal to*

$$p_n(t,s)=\begin{cases}\frac{1}{4\pi}\lim_{\alpha\downarrow 0}\int_\alpha^\infty\left[\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial s^2}\right]m(y,\tau)\,\mathrm{d}y; & \text{if } t,s\notin M\\ 0, & \text{if } t,s\in M\end{cases},$$

$$M=\left\{t,s:\ \frac{1}{n}\operatorname{Tr}\big[(A-I_n(\tau-\rho\theta(0,\tau)))(A-I_n(\tau-\rho\theta(0,\tau)))^*\big]^{-1}<1\right\},$$

and $m(y,\tau)$ *satisfies the canonical equation*

$$\begin{aligned}m(y,\tau)&=\frac{1}{n}\operatorname{Tr}\Bigg[yI_n(1+m(y,\tau))\\&\qquad+\frac{(A-I_n(\tau-\rho\theta(y,\tau)))(A-I_n(\tau-\rho\theta(y,\tau)))^*}{1+m(y,\tau)}\Bigg]^{-1},\qquad(34.1)\\ \theta(y,\tau)&=\frac{1}{2}\int_y^\infty\left(-\frac{\partial}{\partial t}+\mathrm{i}\frac{\partial}{\partial s}\right)m(u,\tau)\,\mathrm{d}u.\end{aligned}$$

This canonical equation has a unique solution in the class of analytic real functions $m(y,\tau),\ y>0$ *in* y.

The V-transform of matrix A is equal to

$$b(\alpha,\tau)=\frac{1}{n}\operatorname{Tr}\left[I\alpha+(A-I\tau)(A-I\tau)^*\right]^{-1},\quad \tau=t+\mathrm{i}s.$$

Using (34.1) we introduce the G_{34}-estimator of $b(\alpha,\tau)$:

$$G_{34}=\frac{m(\hat\beta,\hat z)}{1+m(\hat\beta,\hat z)},$$

where

$$m(\beta,z)=\frac{1}{n}\operatorname{Tr}\left[I\beta+(X-Iz)(X-Iz)^*\right]^{-1},$$

X is an observation of the matrix $A+\Xi$, $\hat\beta$ and $\hat z$ are solutions of equations

$$z-\rho\theta(\beta,z)=\tau,\quad \beta(1+m(\beta,z))^2=\alpha.$$

35. G_{35}-ESTIMATORS OF EIGENVALUES OF RANDOM MATRICES WITH INDEPENDENT PAIRS OF ENTRIES

We call any estimator for eigenvalues of matrix A obtained on the basis of the equation for the boundary of the G-domain of V-density a G_{35}-estimator. Analysing the G-domain of V-density we see a completely different picture of behavior of eigenvalues of random matrices as compared to the results obtained for eigenvalues by the perturbation formulas. For instance, let matrix A in the V-density be symmetric and its eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ satisfy inequalities

$$\alpha_1 > \alpha_2 + c, \ c > 0$$

and

$$n^{-1} \sum_{k=2}^{n} (\alpha_1 - \alpha_k)^{-2} < 1.$$

Let $\lambda(A + \Xi)$ be the eigenvalue of the matrix $A+\Xi$ with the maximal absolute value; then the G_{35}-estimator for eigenvalue α_1 of matrix A is equal to $\lambda(A + \Xi)$. If matrix $A + \Xi$ satisfies the conditions of Theorem 34.1, then

$$p \lim_{n \to \infty} [\operatorname{Re} \lambda(A + \Xi) - \alpha_1] = 0.$$

In this case the norm of random matrix Ξ does not tend to zero. Nevertheless, the random errors in the expression $\operatorname{Re} \lambda(A + \Xi)$ vanish when the dimension of matrix Ξ tends to infinity.

36. G_{36}-ESTIMATOR OF EIGENVECTORS OF MATRICES WITH INDEPENDENT PAIRS OF ENTRIES

We call any estimator for eigenvectors of matrix A obtained with the help the equations for the boundary of the G-domain and for V-density a G_{36} -estimator. Again, analysing the G-domain of V-density we can see another picture of behavior of eigenvectors of random matrices with a comparison of results obtained by the perturbation formulas. Let us assume as above the matrix A in the V-density is symmetric and its eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ satisfy inequalities

$$\alpha_1 > \alpha_2 + c, \ c > 0$$

and

$$n^{-1} \sum_{k=2}^{n} (\alpha_1 - \alpha_k)^{-2} < 1.$$

Let $\vec{\varphi}(A + \Xi)$ be the eigenvector corresponding to the eigenvalue $\lambda(A + \Xi)$ of matrix $A + \Xi$ with the maximal absolute value. Then the G_{36}-estimator of eigenvalue $\vec{\varphi}(A)$ corresponding to the eigenvalue α_1 of matrix A is equal to $\vec{\varphi}(A + \Xi)$. If matrix $A + \Xi$ satisfies the conditions of Theorem 34.1, then

$$p \lim_{n \to \infty} [\operatorname{Re} \vec{\varphi}(A + \Xi) - \vec{\varphi}(A)] = 0$$

The norm of random matrix Ξ does not tend to zero, but the random errors in $\operatorname{Re} \lambda(A + \Xi)$ vanish when the dimension of matrix Ξ tends to infinity. The proof of this result is based on the following formula

$$G_{36} = \oint_{|z+u-\lambda(A+\Xi)|=\delta} \vec{b}^T [A - uI_n + \Xi - zI_n]^{-1} \vec{c} \, \mathrm{d}z = \vec{b}^T [A - uI]^{-1} \vec{c} + \varepsilon_n,$$

where $\delta > 0$ is a certain number.

37. G_{37}-ESTIMATOR OF THE U-STATISTIC

One of the most important problems of the theory of probability and mathematical statistics is the investigation of asymptotic behavior of U-statistics. It confirms the fact that such U-statistics can be used for estimation of functionals of the integral type. There are several directions of investigation of U-statistics. The best known is the so-called Hoeffding's representation. The idea here is to represent U-statistics based on independent observations as the sum of independent random variables and some remainder. The main problem in Hoeffding's method is in the proof that this remainder after certain normalization converges to zero in probability. There are many books and articles dedicated to Hoeffding's method. The second direction is the martingale representation of U-statistics[Gir55]. This representation is more useful and allows us to prove limit theorems for U-statistics in the general case when observations are dependent and may have unbounded moments. Note that these two methods were developed for U-statistics. The third direction is based on the main ideas of general statistical analysis. Namely, we consider the problem of estimating the functionals of integral type under G-condition. In this case, standard U-statistics may have undesirable properties. They are unbiased, but variances of such statistics are very large. Therefore, we will try to find new G-estimators of U-statistics. Consider the functional of the distribution function $F(x)$

$$J = \int \cdots \int f\left(\vec{u}\right) \prod_{k=1}^{m_n} \mathrm{d}F\left(u_k\right), \ \vec{u} = \{u_1, \cdots, u_{m_n}\}^T \in R^{m_n},$$

and suppose that a sequence of independent observations $x_1, \cdots, x_n$ of a random variable with the distribution function $F(x)$ is given. We use the well-known U-statistics of functional J :

$$U_n = \frac{m!\,(n-m)!}{n!} \sum_{\{i_1 < \cdots < i_{m_n}\}} f\left(x_{i_1}, \cdots, x_{i_{m_n}}\right),$$

where $\{i_1 < \cdots < i_{m_n}\}$ is the sample of numbers from numbers $1, \cdots, n$ and this sum is taken over all such samples. U-statistics is unbiased and for fixed m is consistent. But under the G-condition, such statistics in general is not consistent. Let $f\left(x_{i_1}, \cdots, x_{i_{m_n}}\right) \geq c > 0$. Denote

$$V_n = \sum_{\{i_1 < \cdots < i_{m_n}\}} f\left(x_{i_1}, \cdots, x_{i_{m_n}}\right),$$

$$\frac{V_n}{\mathbf{E}\,V_n} = \prod_{k=1}^{n} \frac{\mathbf{E}_{\,k-1} V_n}{\mathbf{E}_{\,k} V_n} = \prod_{k=1}^{n} \left\{1 + \frac{\mathbf{E}_{\,k-1} V_n - \mathbf{E}_{\,k} V_n}{\mathbf{E}_{\,k} V_n}\right\} = \prod_{k=1}^{n} \{1 + \delta_k\},$$

where

$$\delta_k = \frac{\mathbf{E}_{\,k-1} V_n - \mathbf{E}_{\,k} V_n}{\mathbf{E}_{\,k} V_n},$$

$\mathbf{E}_{\,k}$ is the conditional expectation with respect to the fixed random variables $x_{k+1}, \cdots, x_n$.

It is evident that the random variables $\delta_k,\ k = 1, \dots, n$ form martingale differences. Then, using the central limit theorem for the sum of martingale differences, we get

THEOREM 37.1. [Gir55, p. 173–176] *If*

$$p\lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{E}\,_k \delta_k^2 = a,\ a < \infty$$

and for a certain $\varepsilon > 0$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{E}\, |\delta_k|^{2+\varepsilon} = 0,$$

then

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{V_n}{\mathbf{E}\, V_n} < z\right\} = \mathbf{P}\left\{\exp\left[a\eta - \frac{a^2}{2}\right] < z\right\},$$

where η *is the random variable distributed by the standard Normal law* $N(0,1)$.

Consider G-statistics $G_n = bU_n$, where b is a certain number and suppose that for a certain $\rho > 0$

$$\mathbf{E}\, |U_n|^{2+\rho} \le c < \infty.$$

Then by Theorem 37.1 we get

$$\mathbf{E}\, [G_n - J]^2 = (J)^2 \left[b^2 \mathrm{e}^{a^2} - 2b + 1\right]^2 + o(1).$$

This expression is minimal if $b = \mathrm{e}^{-a^2}$. Therefore, for G-estimator $G_n = \mathrm{e}^{-a^2} U_n$ we have

$$\mathbf{E}\, [G_n - J]^2 = (J)^2 \left[1 - \mathrm{e}^{-a^2}\right] + o(1),$$

and for U-statistics we obtain

$$\mathbf{E}\, [U_n - J]^2 = (J)^2 \left[\mathrm{e}^{a^2} - 1\right] + o(1).$$

It is obvious that for large m the G-statistic is much better that the standard U-statistic.

38. G_{38}-ESTIMATORS OF SYMMETRIC FUNCTIONS OF EIGENVALUES OF COVARIANCE MATRICES

The symmetric functions

$$\sum_{i_1 < \cdots < i_l} \lambda_{i_1} \cdots \lambda_{i_l},$$

of eigenvalues λ_k of empirical covariance matrices R_{m_n}, where the sum is over all permutations $i_1 < \cdots < i_l$ of the set $1, 2, \cdots, m_n$ give vise to a very complicated expression. Almost all test statistics, proposed so far, for the commonly encountered hypothesis in multivariate normal theory, visual test of equality of two covariance matrices, MANOVA, and canonical correlation, are of this type. Such functions of fixed order can be calculated on a computer without much difficulty, but for the calculation of symmetric functions of large order, much computer time is needed. Thus, new formulas for symmetric functions, which simplify their calculation, are of great interest.

38.1. Random determinant representation for symmetric function

LEMMA 38.1. [Gir71]

$$S_k := \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k} = \mathbf{E} \det\left[\Xi_{k\times m}\Xi^T_{k\times m}\right](k!)^{-1},$$

where

$$\Xi_{k\times m} = (a_{ij}\xi_{ij})^{j=1,\ldots,m}_{i=1,\ldots,k}, \ \ a_{ij} = \sqrt{\lambda_j},$$

and ξ_{ij} *are independent random variables with* $\mathbf{E}\,\xi_{ij}=0,\ \mathbf{E}\,\xi^2_{ij}=1.$

Proof. It can be shown that

$$\mathbf{E} \det\left[\Xi_{k\times m}\Xi^T_{k\times m}\right] = \sum_{l_1,\cdots,l_k=1}^{m} \det\left[b_{ij}\left(l_i\right)\right]^k_{i,j=1},$$

where $b_{ij}\left(l_i\right) = a_{il_i}\xi_{il_i}a_{jl_i}\xi_{jl_i}$.

Using this equality we get

$$\begin{aligned}
\mathbf{E} \det\left[\Xi_{k\times m}\Xi^T_{k\times m}\right] &= \sum_{l_1,\cdots,l_k=1}^{m} \mathbf{E} \sum_{<i_1,\ldots,i_k>} \pm\left[b_{1i_1}\left(l_{i_1}\right)\times\cdots\times b_{ki_k}\left(l_k\right)\right]\\
&= \sum_{l_1\neq\cdots\neq l_k=1}^{m} \mathbf{E} \det\left[\xi_{il_i}\xi_{jl_i}a_{il_i}a_{jl_i}\right]^k_{i,j=1}\\
&= \sum_{l_1\neq\cdots\neq l_k=1}^{m} \det\left[\mathbf{E}\,\xi_{il_i}\xi_{jl_i}a_{il_i}a_{jl_i}\right]^k_{i,j=1}\\
&= \sum_{l_1\neq\cdots\neq l_k=1}^{m} \det\left[\delta_{ij}a^2_{il_i}\right]^k_{i,j=1} = (k!)\sum_{l_1<\cdots<l_k}\lambda_{l_1}\cdots\lambda_{l_k}.
\end{aligned}$$

Lemma 38.1 is proved.

In this section the method of the random determinant, the invariance principle for symmetric functions of eigenvalues of empirical covariance matrices of large order, and limit theorems for eigenvalues of random matrices are used [Gir1, Gir2, Gir54, Gir71, Gir 84].

THEOREM 38.1 [Gir71, Gir84] *Let* $\alpha_1\leq\cdots\leq\alpha_{m_n}$ *be the eigenvalues of a covariance matrix* R_{m_n} *such that*

$$\inf_n \min_{i=1,\ldots,m_n} \alpha_i > 0,\ \ \sup_n \max_{i=1,\ldots,m_n} \alpha_i < \infty$$

and $\lambda_1\leq\cdots\leq\lambda_{m_n}$ *be the eigenvalues of the empirical covariance matrix* $\hat{R}_{m_n}$,

$$\limsup_{n\to\infty} m_n n^{-1} = \gamma,\ 0<\gamma<1,\ \ \limsup_{n\to\infty} k_m n^{-1} = \beta,\ \ 0<\beta<1.$$

Then

$$\underset{n\to\infty}{p\lim}\left[\frac{1}{m}\ln\sum_{i_1<\cdots<i_{k_m}}\lambda_{i_1}\cdots\lambda_{i_{k_m}}+\frac{\ln k_m!}{m}-\frac{k_m}{m_n}\ln m_n\right.$$
$$\left.+\lim_{A\to\infty}\left(\int_0^A f_n(\alpha)\,d\alpha-\ln A\right)\right]=0,$$

where non-negative function $f_n(\alpha)$ satisfies the equation

$$f_n(\alpha)=\left[\alpha+\int_0^\infty x\,(1+xf_n(\alpha))^{-1}\,\mathrm{d}\mu_n(x)\right]^{-1};\ \ \alpha>0,$$

$\mu_n(x)$ is the normalized spectral function of matrix $\hat{R}_{m_n}$.

Using Theorem 38.1 and the estimator G_2 of the trace of the real resolvent of the covariance matrix we can find a $G_{38}^{(A)}$-estimator of symmetric function

$$\frac{1}{m}\ln\sum_{i_1<\cdots<i_{k_m}}\lambda_{i_1}(R_m)\cdots\lambda_{i_{k_m}}(R_m)$$

of eigenvalues of covariance matrix:

$$G_{38}^{(A)}=\int_0^A g_n(\alpha)\,\mathrm{d}\alpha-\ln\alpha+\frac{\ln k_m!}{m}-\frac{k_m}{m_n}\ln m_n,$$

where $g_n(\alpha)$ is the solution of the equation

$$g_n(\alpha)=\left\{\alpha+\frac{1}{g_n(\alpha)}\left[1-G_2(g_n(\alpha))\right]\right\}^{-1}.$$

39. G_{39}-ESTIMATOR OF SYMMETRIC FUNCTION OF EIGENVALUES OF GRAM RANDOM MATRIX

Using the proof of Theorem 38.1 we can find similar estimators of symmetric functions of eigenvalues of Gram matrices. The spectral theory of random Gram matrices is well developed in Chapter 2. Therefore, repeating the calculations of the estimator G_{38} we find the class of estimators G_{39} of symmetric functions of eigenvalues of different classes of Gram matrices.

40. G_{40} -ESTIMATOR OF PERMANENT OF MATRIX

40.1. The method of random determinants

In contrast to the determinant of a matrix, its permanent is a rather complicated function of the matrix. While a determinant can be calculated on computers without any complications, the calculation of permanents for matrices of order equal to several dozens, using even the most powerful computers, would take hundreds of millennia. Therefore, any formulas for permanents that enable one to simplify the computation are of great interest. In this section the simple analytic relationship of the permanent with the determinant is used ([Gir1], [Gir2], [Gir5], [Gir12], [Gir19], [Gir54], [Gir71], [Gir76], [Gir84]). It gives us the possibility to find the limit values of the permanents, when their orders tend to infinity. The so-called invariance principle for double stochastic matrices has been proved.

The permanent of an $n \times n$ matrix A with entries a_{ij} is defined by

$$\mathrm{per} A = \sum_{<i_1,\ldots,i_n>} a_{1i_1} a_{2i_2} \ldots a_{ni_n},$$

where the sum is over all permutations $< i_1, \ldots, i_n >$ of $(1, 2, \ldots, n)$. The permanent appears in a number of fields, including algebra, combinatorial enumeration and physical sciences, and has been an object of research since its first appearance in 1812 in the work of Cauchy and Binet.

It was shown by the author in [Gir1], [Gir2], [Gir5], [Gir12], [Gir19], [Gir54], [Gir71], [Gir76], [Gir84] that

$$\mathrm{per}\, A = \mathbf{E}\, \det[\sqrt{a_{ij}}\xi_{ij}]^2,$$

or

$$\mathrm{per} A = \mathbf{E} \det[a_{ij}\xi_{ij}] \det[\xi_{ij}],$$

where $\xi_{ij},\ i, j = 1, 2, \ldots$ are independent random variables, $\mathbf{E}\xi_{ij} = 0,\ \mathbf{E}\,\xi_{ij}^2 = 1$, and where by the square root of a complex number we mean its principal value. Therefore, using the Monte Carlo method for statistical estimators of the permanent, we can use the following formula

$$G_{40}^{(s)} = \frac{1}{s} \sum_{k=1}^{s} \det\big[\sqrt{a_{ij}}\xi_{ij}^{(k)}\big]^2,$$

where $\xi_{ij}^{(k)},\ i, j, k = 1, 2, \ldots$ are independent variables, $\mathbf{E}\xi_{ij}^{(k)} = 0,\ \mathbf{E}\xi_{ij}^2 = 1$ and s is any positive integer.

It is easy to see that

$$\mathbf{E}|G_{40}^{(s)} - \mathrm{per}A|^2 = s^{-1}\mathbf{E}\big|\det\left[\sqrt{a_{ij}}\xi_{ij}^{(1)}\right] - \mathrm{per}A\big|^2.$$

Therefore, for any $\varepsilon > 0$, Chebyshev's inequality shows that

$$\mathbf{P}\left\{|G_{40}^{(s)} - \mathrm{per}A| < \varepsilon\right\} \geq 1 - s^{-1}\varepsilon^{-2}\mathbf{E}\left|\det\big[\sqrt{a_{ij}}\xi_{ij}^{(1)}\big] - \mathrm{per}A\right|^2.$$

It is easy to see that with the help of pseudo-random variables and any suitable computer we can find a consistent estimate of a permanent very quickly and without any analytical problems.

41. METHOD OF RANDOM DETERMINANTS. CLASS G_{41}-ESTIMATES OF THE PERMANENT OF A MATRIX

In this part we will not use the random simulation for estimating permanents of a matrix. Instead of simulation we shall find some approximate formulas for the expectation of a random determinant. These formulas will allow us to avoid very cumbersome calculations of the permanent of a matrix on a computer. We call this procedure the *random determinants method.* Here, instead of a nonrandom function of certain variables, we consider expectations of a multidimensional function of random variables, and then, after getting approximate formulas, we simplify this expression.

For convenience sake, we assume that random variables ξ_{ij} have a standard Normal distribution $N(0,1)$.

THEOREM 41.1. [Gir71] *If*

$$\inf_n \min_{i,j=1,\ldots,n} a_{ij} > 0, \quad \sup_n \max_{i,j=1,\ldots,n} a_{ij} < \infty, \tag{41.1}$$

then the G_{41}*-estimate of* $n^{-1}\ln \operatorname{per} A$ *is equal to* $\ln n - \int_0^\infty \left(n^{-1}\sum_{k=1}^n c_k(\alpha) - \alpha^{-1}\chi(\alpha > 1)\right) \mathrm{d}\alpha$ *and*

$$\lim_{n\to\infty}\left[n^{-1}\ln \operatorname{per} A - \ln n + \int_0^\infty \left(n^{-1}\sum_{k=1}^n c_k(\alpha) - \alpha^{-1}\chi(\alpha > 1)\right)\mathrm{d}\alpha\right] = 0, \tag{41.2}$$

or

$$\lim_{n\to\infty}\left\{n^{-1}\ln \operatorname{per} A - \ln n + \lim_{A\to\infty}\left[\int_0^A \left(n^{-1}\sum_{k=1}^n c_k(\alpha)\right)\mathrm{d}\alpha - \ln A\right]\right\} = 0,$$

where positive real functions $c_j(\alpha)$; $j = 1,\ldots,n$ *satisfy the system of equations*

$$c_k(\alpha) = \left[\alpha + \sum_{l=1}^n n^{-1}a_{kl}\left(1 + \sum_{j=1}^n n^{-1}a_{jl}c_j(\alpha)\right)^{-1}\right]^{-1}; \ \alpha > 0; \quad k = 1,\ldots,n. \tag{41.3}$$

This equation has a unique solution in the class L *of positive real analytic functions* $c_j(\alpha)$ *for* $\alpha > 0$.

THEOREM 41.2. *If the condition (41.1) is fulfilled, then the* G_{41}*-estimate of* $n^{-1}\ln \operatorname{per} A$ *is equal to* $\ln n + \int_0^\infty \left(n^{-1}\sum_{k=1}^n p_k(x)\right)\ln x \,\mathrm{d}x$ *and*

$$\lim_{n\to\infty}\left[n^{-1}\ln \operatorname{per} A - \ln n - \int_0^\infty \left(n^{-1}\sum_{k=1}^n p_k(x)\right)\ln x\,\mathrm{d}x\right] = 0, \tag{41.4}$$

where $p_k(x)$ *are distribution densities whose Stieltjes transforms*

$$c_{kn}(z) = \int_0^\infty (x - z)^{-1} p_k(x)\,\mathrm{d}x, \quad z = t + \mathrm{i}s, \ s \neq 0 \tag{41.5}$$

satisfy the system of equations

$$c_{kn}(z) = \left[\sum_{l=1}^n n^{-1}a_{kl}\left(1 + \sum_{p=1}^n n^{-1}a_{pl}c_{pn}(z)\right)^{-1} - z\right]^{-1}, \ k = 1,\ldots,n \tag{41.6}$$

which has a unique solution in the class of analytic functions: $\operatorname{Im} c_{jn}(z) > 0; \operatorname{Im} z > 0$. *The functions* $p_k(x)$; $k = 1,\ldots,n$ *are imaginary parts of solutions* $m_j(x)$ *of the system of equations*

$$m_k(x) = \left[-x + \sum_{l=1}^n n^{-1}a_{kl}\left(1 + \sum_{j=1}^n n^{-1}a_{jl}m_j(x)\right)^{-1}\right]^{-1}; \ x > 0; \ k = 1,\ldots,n \tag{41.7}$$

where $m_j(x) = g_j(x) + \mathrm{i}\pi p_j(x)$. The system of equations (41.7) has a unique solution in the class of functions $\{g_k(x);\ p_k(x) > 0;\ x > 0;\ k = 1, \dots, n\}$.

42. G_{42}-ESTIMATOR OF A PRODUCT OF MATRICES

One of the most important problems of applied mathematics is investigation of the asymptotic behavior of distribution of the product of random matrices

$$\prod_{k=1}^{n} X_{m\times m}^{(k)},$$

where $X_{m\times m}^{(k)} = A_{m\times m}^{(k)} + \Xi_{m\times m}^{(k)}$ are independent observations, where $A_{m\times m}^{(k)}$ are unknown deterministic matrices and $\Xi_{m\times m}^{(k)},\ k = 1, \dots, n$ are certain independent random matrices. Such product of random matrices can be used for estimation of the solution of the system of differential equations with random coefficients. We mention several directions of investigation of distribution of the product of random matrices. The best known method applies to the case when matrices belong to a compact group. A second direction was developed for investigating matrices that belong to a certain locally compact group. And a third direction is connected with limit theorems for the product of random matrices in the scheme of series, when every matrix converges on probability to a nonrandom matrix when the number increases. In Girko's article [Gir21], the martingale representation of the product of random matrices was considered and a limit theorem was proved. This representation allows us to prove limit theorems for the product of random matrices in the general case when observations are dependent and may have unbounded moments. Consider the simplest functional of the product of random matrices:

$$V_n = \left\{ \prod_{k=1}^{n} X_{m\times m}^{(k)} \right\}_{pl},$$

where $X_{m\times m}^{(k)} = A_{m\times m}^{(k)} + \Xi_{m\times m}^{(k)}$ are independent observations, matrices $A_{m\times m}^{(k)}$ are unknown and $\Xi_{m\times m}^{(k)}$, $k = 1, \dots, n$ are certain independent random matrices such that $\mathbf{E}\,\Xi_{m\times m}^{(k)} = \{0\}_{m\times m}$, $k = 1, \dots, n$ and p and l any fixed number of entries of matrix $\prod_{k=1}^{n} X_{m\times m}^{(k)}$.

Let $\left\{ \prod_{k=1}^{n} A_{m\times m}^{(k)} \right\}_{pl} \neq 0$. Denote

$$\frac{V_n}{\mathbf{E}\,V_n} = \prod_{k=1}^{n} \frac{\mathbf{E}_{\,k-1} V_n}{\mathbf{E}_{\,k} V_n} = \prod_{k=1}^{n} \left\{ 1 + \frac{\mathbf{E}_{\,k-1} V_n - \mathbf{E}_{\,k} V_n}{\mathbf{E}_{\,k} V_n} \right\} = \prod_{k=1}^{n} \{1 + \delta_k\},$$

where

$$\delta_k = \frac{\mathbf{E}_{\,k-1} V_n - \mathbf{E}_{\,k} V_n}{\mathbf{E}_{\,k} V_n},$$

with $\mathbf{E}_{\,k}$ being the conditional expectation with respect to the fixed random matrices $X_{k+1}, \cdots, X_n$. It is evident that random variables $\delta_k,\ k = 1, \dots, n$ form martingale differences. Then, using the central limit theorem for the sum of martingale differences we get

Theorem 42.1. *If*

$$p\lim_{n\to\infty}\sum_{k=1}^{n}\mathbf{E}_k\delta_k^2 = a,\ \ a<\infty$$

and for a certain $\varepsilon>0$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbf{E}\left|\delta_k\right|^{2+\varepsilon}=0,$$

then

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{V_n}{\mathbf{E}\,V_n}<z\right\}=\mathbf{P}\left\{\exp\left[a\eta-\frac{a^2}{2}\right]<z\right\},$$

where η *is the random variable distributed by the standard Normal law* $N\left(0,1\right)$.

Consider G_{42}-statistic $G_n = b\left\{\prod_{k=1}^{n}X_{m\times m}^{(k)}\right\}_{pl}$, where b is a certain number, and suppose that for a certain $\rho>0$

$$\mathbf{E}\left|V_n\right|^{2+\rho}\le c<\infty.$$

Then by Theorem 42.1 we get

$$\mathbf{E}\left[G_n-J\right]^2=(J)^2\left[b^2\mathrm{e}^{a^2}-2b+1\right]^2+o\left(1\right),$$

where

$$J=\left\{\prod_{k=1}^{n}A_{m\times m}^{(k)}\right\}_{pl}.$$

This expression will be minimal if $b=\mathrm{e}^{-a^2}$. Therefore, for G-estimator $G_{42} = b\left\{\prod_{k=1}^{n}X_{m\times m}^{(k)}\right\}_{pl}$ we have

$$\mathbf{E}\left[G_{42}-J\right]^2=(J)^2\left[1-\mathrm{e}^{-a^2}\right]+o\left(1\right),$$

and for the standard estimator we obtain

$$\mathbf{E}\left[\prod_{k=1}^{n}X_{m\times m}^{(k)}-J\right]^2=(J)^2\left[\mathrm{e}^{a^2}-1\right]+o\left(1\right).$$

It is obvious that for large a, the G_{42}-statistics is much better than the standard statistics.

43. G_{43}-ESTIMATOR OF PRODUCT OF MATRICES IN THE SCHEME OF SERIES

Consider the product of random matrices in the scheme of series

$$\prod_{k=1}^{n}\left[I_{m_n}+X_{m_n}^{(k)}\right],$$

where

$$X_{m_n}^{(k)}=A_{m_n}^{(k)}+\Xi_{m_n}^{(k)}$$

are independent observations, $A_{m_n}^{(k)}$ are unknown nonrandom matrices and $\Xi_{m_n}^{(k)}$, $k = 1, \ldots, n$ are certain independent random matrices.

Consider the simplest functional of the product of random matrices:

$$V_n = \left\{ \prod_{k=1}^{n} \left[I_{m_n} + X_{m_n}^{(k)} \right] \right\}_{pl},$$

where $\mathbf{E}\, \Xi_{m_n}^{(k)} = \{0\}_{m_n}$, $k = 1, \ldots, n$ and p and l any fixed number of entries of the matrix $\prod_{k=1}^{n} \left[I_{m_n} + X_{m_n}^{(k)} \right]$.

Let

$$\left\{ \prod_{k=1}^{n} \left[I_{m_n} + X_{m_n}^{(k)} \right] \right\}_{pl} \neq 0.$$

Denote

$$\frac{V_n}{\mathbf{E}\, V_n} = \prod_{k=1}^{n} \frac{\mathbf{E}_{\,k-1} V_n}{\mathbf{E}_{\,k} V_n} = \prod_{k=1}^{n} \left\{ 1 + \frac{\mathbf{E}_{\,k-1} V_n - \mathbf{E}_{\,k} V_n}{\mathbf{E}_{\,k} V_n} \right\} = \prod_{k=1}^{n} \{1 + \delta_k\},$$

where

$$\delta_k = \frac{\mathbf{E}_{\,k-1} V_n - \mathbf{E}_{\,k} V_n}{\mathbf{E}_{\,k} V_n},$$

$\mathbf{E}_{\,k}$ is the conditional expectation with respect to the fixed random matrices $X_{m_n}^{(k+1)}, \cdots, X_{m_n}^{(n)}$.

It is evident that random variables δ_k, $k = 1, \ldots, n$ form martingale differences. Then, using the central limit theorem for the sum of martingale differences we get, as in previous section, for G-statistics $G_{43} = b \left\{ \prod_{k=1}^{n} X_{m \times m}^{(k)} \right\}_{pl}$, where $b = \mathrm{e}^{-a^2}$, a is a certain number and

$$\mathbf{E}\, [G_{43} - J]^2 = (J)^2 \left[1 - \mathrm{e}^{-a^2} \right] + o\,(1)$$

We obtain for the standard estimator

$$\mathbf{E} \left[\prod_{k=1}^{n} X_{m \times m}^{(k)} - J \right]^2 = (J)^2 \left[\mathrm{e}^{a^2} - 1 \right] + o\,(1)\,.$$

It is obvious that for large a, the G_{43}-statistic is much better than the standard statistics $V_n = \left\{ \prod_{k=1}^{n} \left[I_{m_n} + X_{m_n}^{(k)} \right] \right\}_{pl}$.

44. CLASS OF G_{44}-ESTIMATORS OF SOLUTIONS OF THE SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH COVARIANCE MATRIX OF COEFFICIENTS

In this section we consider a system of linear differential equations with random coefficients

$$\frac{\mathrm{d}\vec{x}\,(t)}{\mathrm{d}t} = \Xi \vec{x}\,(t)\,, \quad 0 \le t \le T, \quad \vec{x}\,(0) = \vec{c}, \tag{44.1}$$

where $\Xi = (\xi_{ij})_{i,j=1}^{n}$ is a random symmetric real matrix of the order n, $\vec{x}^T(t) = \{x_1(t), \cdots, x_n(t)\}$ and the dimension of such a system is large and every random coefficient tends to a certain constant in probability when the dimension of this system tends to infinity, i.e., no single coefficient is influential enough to dominate the system as a whole. The self-averaging of the solutions of a system of linear differential equations with random coefficients means that the vector-solution $\vec{x}(t)$ converges to the solution of a certain nonrandom equation when the dimension of system (44.1) tends to infinity. It is well known that the solution of system (44.1) is equal to

$$\vec{x}(t) = \exp\{t\Xi\}\vec{c}, \;\; 0 \le t \le T.$$

The need to solve such systems arises in different problems of calculus, differential and integral equations, experimental design, etc.

Unfortunately, in practical problems, it is very difficult to find the distribution functions of random coefficients ξ_{ij} of such systems. For this reason, we have developed a new method of analysis in which these coefficients ξ_{ij} have an arbitrary distribution function. It is natural in this case to use the methods of General Statistical Analysis.

44.1 Formulation of the problem under the conditions of general statistical analysis

The system

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = \left\{n^{-1}\sum_{k=1}^{n} X_i\right\}\vec{x}(t), \;\; 0 \le t \le T$$

with random coefficients arises when, instead of a matrix of coefficients Ξ, we have n observations X_i of random matrices Ξ. Moreover, replacing a nonlinear system $\frac{\mathrm{d}\vec{y}(t)}{\mathrm{d}t} = \vec{f}(\vec{y}(t))$ by a system of linear differential equations, we arrive at a system of large dimension. This example clearly shows that we must solve this problem within the framework of general statistical analysis because both parameters, number of observations, and the number of interval partitions for integration and differentiation tend to infinity. Besides, we cannot choose these two parameters arbitrarily large, because the growth of the first parameter leads to large computer time. The growth of the second parameter leads to large losses in energy and material resources, and sometimes it is not possible to increase these parameters. Suppose that the following system is given:

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = R_n\vec{x}(t), \;\; 0 \le t \le T, \;\; \vec{x}(0) = \vec{c}, \tag{44.2}$$

where R_n is a covariance matrix. The problem is to find a G-estimator of the expression $\vec{b}^T\vec{x}(t) = \vec{b}^T\exp\{tR_n\}\vec{c}, \;\; 0 \le t \le T$, when the empirical covariance matrix $\hat{R}_n$ is given, and $\vec{b}$ is a n-dimensional vector. Using estimators G_2 and G_{25} we find the G_{44}-estimator of the value $\vec{b}^T\vec{x}(t) = \vec{b}^T\exp\{tR_n\}\vec{c}, \;\; 0 \le t \le T$:

$$G_{44} = (2\pi\mathrm{i})^{-1}\vec{b}\oint_{|z|=\delta}\exp(tz)\frac{\hat{\theta}(z)}{z}\left[\hat{R}_n - \hat{\theta}(z)\right]^{-1}\vec{c}\,\mathrm{d}z,$$

where the integral is taken around the unit circle, $\delta > G_{26}^{\max}$, and $\hat{\theta}(z)$ is the measurable solution of the equation

$$\hat{\theta}(z)\frac{1}{n}\mathrm{Tr}\left\{\hat{R}_{m_n}-\hat{\theta}(z)I_{m_n}\right\}^{-1}-\left(1-\frac{m_n}{n}\right)+\frac{\hat{\theta}(z)}{z}=0.$$

THEOREM 44.1. *Suppose $\vec{x}_1,\cdots,\vec{x}_n$ are the sample of independent observations of a random vector,*

$$\vec{x}_k=R_{m_n}^{1/2}\vec{\xi}_k+\vec{a},\ \ \mathbf{E}\,\vec{\xi}_k=0,\ \mathbf{E}\,\vec{\xi}_k\vec{\xi}_k^T=I_{m_n},\ \ \vec{\xi}_k^T=\{\xi_{ik},i=1,...,m_n\},$$

random variables ξ_{ik} are independent and for some $\beta>0$

$$\mathbf{E}\,|\xi_{ij}\sqrt{n}|^{4+\delta}\leq c<\infty,$$

$$\lambda_i(R_{m_n})<c<\infty,\ \ i=1,...,m_n,$$

$$\liminf_{n\to\infty}m_nn^{-1}>0,\ \limsup_{n\to\infty}m_nn^{-1}<\infty$$

$$\vec{b}^T\vec{b}+\vec{c}^T\vec{c}\leq c<\infty,$$

then with probability one for every $t>0$ and $T>0$

$$\lim_{n\to\infty}\left\{G_{44}-\vec{b}^T\exp(tR_n)\,\vec{c}\right\}=0,\ \ 0\leq t\leq T.$$

45. CLASS OF G_{45}-ESTIMATORS OF SOLUTIONS OF THE SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH NON-NEGATIVE DEFINED MATRIX OF COEFFICIENTS

Similarly we can find an estimator for the case when the matrix of the coefficients of system (44.1) is AA^T, where A is a certain nonrandom matrix.

Suppose that

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t}=AA^T\vec{x}(t),\ \ 0\leq t\leq T,\ \ \vec{x}(0)=\vec{c},$$

where A is a triangular matrix and the problem is to find the G-estimator of the expression $\vec{b}^T\vec{x}(t)=\vec{b}^T\exp\left\{tAA^T\right\}\vec{c},\ \ 0\leq t\leq T$, when observation Ξ of matrix $A+H$ is given, and $\vec{b}$ is an n-dimensional vector. Using estimators G_2 and G_{29} we find the G_{45}-estimator of the value $\vec{b}^T\vec{x}(t)=\vec{b}^T\exp\left\{tAA^T\right\}\vec{c},\ \ 0\leq t\leq T$:

$$G_{45}=(2\pi\mathrm{i})^{-1}\,\vec{b}\oint\limits_{|z|=\delta}\exp(tz)\,\varphi\left(\hat{\theta}(z),\Xi\Xi^T\right)\left[1+\gamma\varphi\left(\hat{\theta}(z),\Xi\Xi^T\right)\right]^{-1}\vec{c}\,\mathrm{d}z,$$

where $\varphi\left(z,AA^T\right)=m_n^{-1}\mathrm{Tr}\left[AA^T-zI_{m_n}\right]^{-1}$, $\delta>G_{29}^{\max}$, and $\hat{\theta}(z)$ is the measurable solution of the equation

$$-\hat{\theta}(z)\left\{1+\frac{1}{n}\mathrm{Tr}\left[\Xi\Xi^T-\hat{\theta}(z)I_{m_n}\right]^{-1}\right\}^2$$
$$+\left(1-\frac{m_n}{n}\right)\left\{1+\frac{1}{n}\mathrm{Tr}\left[\Xi\Xi^T-\hat{\theta}(z)I_{m_n}\right]^{-1}\right\}=-z$$

Ξ is an observation of the matrix $A+H$, H is a certain random matrix.

THEOREM 45.1. *If, for every n, the entries $\xi_{ij}^{(n)}$, $i=1,\ldots,m_n$; $j=1,\ldots,n$ of random matrix Ξ are independent, $\mathbf{E}\,\xi_{ij}^{(n)}=a_{ij}^{(n)}$, $\mathbf{Var}\,\xi_{ij}^{(n)}=n^{-1}$; for a certain $\delta>0$*

$$\mathbf{E}\left|(\xi_{ij}^{(n)}-a_{ij}^{(n)})\sqrt{n}\right|^{2+\delta}\le c_1<\infty,\quad \max_{i=1,\ldots,m}\sum_{j=1}^{n}a_{ij}^2\le c_2<\infty,$$
$$0<\liminf_{n\to\infty}\frac{m_n}{n}<\limsup_{n\to\infty}\frac{m_n}{n}<1,$$
$$\vec{b}^T\vec{b}+\vec{c}^T\vec{c}\le c<\infty,$$

then with probability one for every $t>0$

$$\lim_{n\to\infty}\left\{G_{45}-\vec{b}^T\exp\left(tAA^T\right)\vec{c}\right\}=0,\ \ 0\le t\le T.$$

46. G_{46}-ESTIMATOR FOR SOLUTION OF THE SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH SYMMETRICAL MATRIX OF COEFFICIENTS

Suppose that as before

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t}=A_n\vec{x}(t),\ \ 0\le t\le T,\ \ \vec{x}(0)=\vec{c}, \tag{46.1}$$

where A_n is a symmetric matrix and the problem is to find the G-estimator of the expression $\vec{b}^T\vec{x}(t)=\vec{b}^T\exp\{tA\}\vec{c}$, $0\le t\le T$, when observation matrix $X_n=A_n+\Xi_n$ is given, and $\vec{b}$ is an n-dimensional vector. Using estimators G_{31} and G_{32} we find the G_{46}-estimator of the value $\vec{b}^T\vec{x}(t)=\vec{b}^T\exp\{tA_n\}\vec{c}$, $0\le t\le T$:

$$G_{46}=(2\pi\mathrm{i})^{-1}\vec{b}\oint_{|z|=\delta}\exp(tz)\left[X_n-\hat{\theta}(z)\right]^{-1}\vec{c}\,\mathrm{d}z,$$

where $\delta>G_{32}^{\max}$, $\hat{\theta}(z)$ is the measurable solution of the G_{31} equation

$$\hat{\theta}(z)+n^{-1}\mathrm{Tr}\left[\Xi-\hat{\theta}(z)I_n\right]^{-1}=z.$$

THEOREM 46.1. *If, for every n, random entries $\xi_{ij}^{(n)}$, $i\ge j$, $i,j=1,\ldots,n$ are such that $\mathbf{E}\,(\xi_{ij}^{(n)})^2=n^{-1}$, for some $\delta>0$*

$$\mathbf{E}\,|\xi_{ij}\sqrt{n}|^{4+\delta}\le c<\infty,\quad \max_{i=1,\ldots,n}\sum_{j=1}^{n}a_{ij}^2\le c<\infty,\quad \vec{b}^T\vec{b}+\vec{c}^T\vec{c}\le c<\infty,$$

then with probability one for every $t>0$

$$\lim_{n\to\infty}\left\{G_{46}-\vec{b}^T\exp(tA_n)\vec{c}\right\}=0,\ \ 0\le t\le T.$$

47. G_{47}-ESTIMATOR FOR SOLUTION OF THE SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

In ths section we consider an arbitrary matrix of coefficients of system (44.1). As in the previous two sections we use the Cauchy integral formula

$$f(a) = (2\pi \mathrm{i})^{-1} \oint_{\Gamma} f(z)(z-a)^{-1}\, \mathrm{d}z, \tag{47.1}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}a^n} f(a)(n!)^{-1} = (2\pi \mathrm{i})^{-1} \oint_{\Gamma} f(z)(z-a)^{-n-1}\, \mathrm{d}z;\;\; n = 1, 2, ...,$$

where $f(z)$ is an analytic function, a is inside a circle Γ, which is positively oriented, and the integral is taken around the unit circle.

47.1. V_1-Transform of the solution of the system of linear differential equations

Denote $R = (\Xi - zI)^{-1}$. Using formula (44.1) for the solution of the system of equation

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = \Xi \vec{x}(t),\;\; \vec{x}(0) = \vec{c},\;\; 0 \le t \le T,$$

we have

$$\vec{a}^T \vec{x}(t) = \vec{a}^T \exp\{t\Xi\}\, \vec{c} = -(2\pi \mathrm{i})^{-1} \oint_{\Gamma} \mathrm{e}^{tz} \vec{a}^T R(z)\, \vec{c}\, \mathrm{d}z, \tag{47.2}$$

where G is a positively oriented circle containing all eigenvalues of matrix X, $\vec{a}^T = \{a_1, \cdots, a_n\}$ is an arbitrary vector.

Using the integral representation for solutions of SLAE (see [Gir84]) we prove

LEMMA 47.1. (V_1-Transform of the solution of the system of linear differential equations)

$$\vec{a}^T \vec{x}(t) = -\frac{1}{4\pi \mathrm{i}} \lim_{\alpha \downarrow 0} \frac{\partial}{\partial \gamma} \oint_{\Gamma} \mathrm{e}^{tz} \Big\{ \ln\det\Big[\big(\Xi - zI + \gamma \vec{c}\vec{a}^T\big)\big(\Xi - zI + \gamma \vec{c}\vec{a}^T\big)^* + \alpha I \Big]$$

$$-\mathrm{i} \ln\det\Big[\big(\Xi - zI + \mathrm{i}\gamma \vec{c}\vec{a}^T\big)\big(\Xi - zI + \mathrm{i}\gamma \vec{c}\vec{a}^T\big)^* + \alpha I \Big] \Big\}_{\gamma=0} \mathrm{d}z. \tag{47.3}$$

47.2. V_2-Transform of the solution of the system of linear differential equations

Using (47.3) and the integral representation for solutions of SLAE (see [Gir84]) we get

LEMMA 47.2. (V_2-Transform of the solution of the system of linear differential equations)

$$\begin{aligned} \vec{a}^T \vec{x}(t) &= \vec{a}^T \exp\{t\Xi\}\, \vec{c} = (2\pi \mathrm{i})^{-1} \oint_{\Gamma} \mathrm{e}^{tz} \vec{a}^T (zI - \Xi)^{-1} \vec{c}\, \mathrm{d}z \\ &= -\lim_{\alpha \downarrow 0} (2\pi \mathrm{i})^{-1} \oint_{\Gamma} \mathrm{e}^{tz} \frac{1}{2} \int_{\alpha}^{\infty} \frac{\partial}{\partial \gamma} \mathrm{Tr}\, [Q(y,\gamma) - \mathrm{i} Q(y, \mathrm{i}\gamma)]_{\gamma=0}\, \mathrm{d}y\, \mathrm{d}z, \end{aligned} \tag{47.4}$$

where

$$Q(y,\gamma) = \left\{ yI + \left(\Xi - zI + \gamma \vec{c}\vec{a}^T\right)\left(\Xi - zI + \gamma \vec{c}\vec{a}^T\right)^* \right\}^{-1}.$$

47.3. V_3-Transform of the solution of the system of linear differential equations

When random entries of a matrix have different variances, we will use the differential representation for solutions of SLAE (see Chapter 6).

LEMMA 47.3. (V_3-Transform of the solution of the system of linear differential equations). *If* $\vec{a}^T\vec{a}^T \geq c > 0$, *then*

$$\begin{aligned}\vec{a}^T\vec{x}(t) &= \vec{a}^T \exp\{t\Xi\}\vec{c} \\ &= \lim_{\alpha\downarrow 0}(2\pi \mathrm{i})^{-1}\oint_{\Gamma} \mathrm{e}^{tz}\frac{\partial}{\partial\gamma}\left[\frac{\left[\vec{a}^T G(\alpha,\gamma)\vec{a}\right]}{2\vec{a}^T G(\alpha,0)\vec{a}} - \mathrm{i}\frac{\left[\vec{a}^T G(\alpha,\mathrm{i}\gamma)\vec{a}\right]}{2\vec{a}^T G(\alpha,0)\vec{a}}\right]\Bigg|_{\gamma=0}\mathrm{d}z,\end{aligned} \tag{47.5}$$

where

$$G(\alpha,\gamma) = \left\{ \alpha I + \left(\Xi - zI + \gamma \vec{c}\vec{a}^T\right)^*\left(\Xi - zI + \gamma \vec{c}\vec{a}^T\right) \right\}^{-1}.$$

We call the expression

$$\vec{a}^T\vec{V}_{1\alpha}(t) = -\frac{1}{4\pi\mathrm{i}}\oint_{\Gamma}\mathrm{e}^{tz}\frac{\partial}{\partial\gamma}\left\{\ln\det Q(\alpha,\gamma) - \mathrm{i}\ln\det Q(\alpha,\mathrm{i}\gamma)\right\}_{\gamma=0}\mathrm{d}z$$

a linear form of the V_1-regularized solution of the system of linear differential equations, and

$$\vec{a}^T\vec{V}_{2\alpha}(t) = -\frac{1}{4\pi\mathrm{i}}\oint_{\Gamma}\mathrm{e}^{tz}\left\{\frac{1}{2}\int_{\alpha}^{\infty}\frac{\partial}{\partial\gamma}\mathrm{Tr}\,\left[Q(y,\gamma) - \mathrm{i}Q(y,\mathrm{i}\gamma)\right]_{\gamma=0}\mathrm{d}y\right\}\mathrm{d}z$$

a linear form of the V_2-regularized solution of the system of linear differential equations and

$$\vec{a}^T\vec{V}_{3\alpha}(t) = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma}\mathrm{e}^{tz}\left\{\frac{\partial}{\partial\gamma}\left[\frac{\left[\vec{a}^T G(\alpha,\gamma)\vec{a}\right]}{4\vec{a}^T G(\alpha,0)\vec{a}} - i\frac{\left[\vec{a}^T G(\alpha,\mathrm{i}\gamma)\vec{a}\right]}{4\vec{a}^T G(\alpha,0)\vec{a}}\right]\right\}_{\gamma=0}\mathrm{d}z$$

a linear form of the V_3-regularized solution of the system of linear differential equations, where $\alpha > 0$.

47.4. Limit theorem for singular values of random complex matrices

It follows from [Gir84]

THEOREM 47.1. [Gir84] *Let* Ξ *be a random complex matrix of the size* $n \times n$, *whose entries* $\xi_{ij}^{(n)}$; $i = 1,\dots,n$; $j = 1,\dots,n$ *are independent for every* n *and are defined on common probability space,*

$$\mathbf{E}\,\Xi = A = (a_{ij})_{i=1,\dots,n}^{j=1,\dots,n}\,\mathbf{E}\,\xi_{kp}^{(n)} = a_{kp}^{(n)};$$

$$\mathbf{E}\left|\xi_{kp}^{(n)} - a_{kp}^{(n)}\right|^2 = n^{-1}; \quad k = 1, ..., n; \quad p = 1, ..., n,$$

for a certain $\delta > 0$

$$\sup_n \max_{\substack{i=1,\ldots,n;\\ j=1,\ldots,n}} \mathbf{E}\left|\left(\xi_{ij}^{(n)} - \mathbf{E}\,\xi_{ij}^{(n)}\right)\sqrt{n}\right|^{4+\delta} < \infty,$$

$$\sup_n \max_{k=1,\ldots,n} \sum_{j=1}^{n}\left[|a_{kj}|^2 + |a_{jk}|^2\right] < \infty,$$

$$\alpha_n\left(AA^*\right) \leq c_1 < \infty,$$

$$\beta_N = \max_{v_i}\left\{v_i\left[1 - n^{-1}\sum_{k=1}^{n}\frac{1}{(\alpha_k - v_i)}\right]^2\right\},$$

and v_i *are the real solutions of the* L_2 *equation*

$$1 - n^{-1}\sum_{k=1}^{n}\frac{1}{(\alpha_k - v_i)} = 2v_i n^{-1}\sum_{k=1}^{n}\frac{1}{(\alpha_k - v_i)^2},$$

where

$$\alpha_1\left(AA^*\right) \leq \cdots \leq \alpha_n\left(AA^*\right)$$

are the singular values of matrix A.

Then with probability one

$$\lim_{n\to\infty}\left[\lambda_{\max}\left(\Xi\Xi^*\right) - \beta_N\right] = 0; \;\; \beta_N \leq c < \infty,$$

where c *is a certain constant.*

Thus, we see that in the most interesting case for random matrices the absolute values of their eigenvalues are bounded in probability. This result allows us to use limit theorems for the solutions of the system of linear differential equations with random coefficients.

47.5. Limit theorem for V-transforms of the solution of the system of linear differential equations

THEOREM 47.2. *If, in addition to the conditions of Theorem 47.1,* $\vec{a}^T\vec{a} \geq c > 0$, *and* $\limsup_{n\to\infty}\left[\vec{a}^T\vec{a} + \vec{c}^T\vec{c}\right] < \infty$, *then with probability one*

$$\lim_{\alpha\downarrow 0}\limsup_{n\to\infty}\sup_{0\leq t\leq T}\left|\vec{a}^T \mathrm{e}^{t\Xi}\vec{b} + \oint_{\Gamma:|z|^2>\beta_N}\frac{\mathrm{e}^{tz}}{4\pi\mathrm{i}} \times\frac{\partial}{\partial\gamma}\left\{\ln\det Q\left(\alpha,\gamma\right) - \mathrm{i}\ln\det Q\left(\alpha,\mathrm{i}\gamma\right)\right\}_{\gamma=0}\mathrm{d}z\right| = 0,$$

$$\lim_{\alpha\downarrow 0}\limsup_{n\to\infty}\sup_{0\le t\le T}\Bigg|\vec{a}^T \mathrm{e}^{t\Xi}\vec{b} + \oint\limits_{\Gamma:|z|^2>\beta_N} \frac{\mathrm{e}^{tz}}{4\pi \mathrm{i}}$$

$$\times \frac{1}{2}\int_{\alpha}^{\infty}\frac{\partial}{\partial\gamma}\mathrm{Tr}\,\left[Q\left(y,\gamma\right) - \mathrm{i}Q\left(y,\mathrm{i}\gamma\right)\right]_{\gamma=0}\mathrm{d}z\Bigg| = 0,$$

$$\lim_{\alpha\downarrow 0}\limsup_{n\to\infty}\sup_{0\le t\le T}\Bigg|\vec{a}^T \mathrm{e}^{t\Xi}\vec{b} - \oint\limits_{\Gamma:|z|^2>\beta_N} \frac{\mathrm{e}^{tz}}{4\pi \mathrm{i}}$$

$$\times \frac{\partial}{\partial\gamma}\left[\frac{\left[\vec{a}^T G\left(\alpha,\gamma\right)\vec{a}\right]}{\vec{a}^T G\left(\alpha,0\right)\vec{a}} - \mathrm{i}\frac{\left[\vec{a}^T G\left(\alpha,\mathrm{i}\gamma\right)\vec{a}\right]}{\vec{a}^T G\left(\alpha,0\right)\vec{a}}\right]_{\gamma=0}\mathrm{d}z\Bigg| = 0,$$

where

$$Q\left(y,\gamma\right) = \left\{yI + \left(\Xi - zI + \gamma\vec{b}\vec{a}^T\right)\left(\Xi - zI + \gamma\vec{b}\vec{a}^T\right)^*\right\}^{-1},$$

$$G\left(\alpha,\gamma\right) = \left\{\alpha I + \left(\Xi - zI + \gamma\vec{c}\vec{a}^T\right)^*\left(\Xi - zI + \gamma\vec{c}\vec{a}^T\right)\right\}^{-1}.$$

Consider G-estimator: $G_{47}\left(t\right) = \vec{a}^T\exp\left\{tX_n\right\}\vec{c}$, $X_n = A_n + \Xi_n$. From [Gir84] it follows

THEOREM 47.3. *Let the random entries $\xi_{ij}^{(n)}, i,j = 1,\ldots,n$ of real matrix Ξ be independent for every n,*

$$\mathbf{E}\,\xi_{ij}^{(n)} = 0, \quad \mathbf{Var}\xi_{ij}^{(n)} = n^{-1}, \quad \vec{a}^T\vec{a} \ge c > 0,$$

where c is some constant,

$$\sup_n \max_{i,j=1,\ldots,n}\left\{\sum_{j=1}^{n}\left[\left|a_{ij}^{(n)}\right| + |c_j|\right] + \sum_{i=1}^{n}\left[\left|a_{ij}^{(n)}\right| + |a_i|\right]\right\} < \infty,$$

and let Lindeberg's condition be fulfilled: for every $\tau > 0$

$$\lim_{n\to\infty}\max_{i,j=1,\ldots,n}\left[\sum_{j=1}^{n}\mathbf{E}\left[\xi_{ij}^{(n)}\right]^2\chi\left\{\left|\xi_{ij}^{(n)}\right| > \tau\right\}\right.$$

$$\left. + \sum_{i=1}^{n}\mathbf{E}\left[\xi_{ij}^{(n)}\right]^2\chi\left\{\left|\xi_{ij}^{(n)}\right| > \tau\right\}\right] = 0,$$

where χ is the indicator of a random event. Then for any $t > 0$

$$p\lim_{n\to\infty}\left[G_{47}\left(t\right) - \vec{a}^T\exp\left\{tA\right\}\vec{c}\right] = 0.$$

48. G_{48}**-ESTIMATOR FOR SOLUTION OF THE SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WHEN COEFFICIENTS HAVE ARBITRARY VARIANCES**

Suppose that

$$\frac{\mathrm{d}\vec{x}\,(t)}{\mathrm{d}t} = A_n\vec{x}\,(t)\,,\ \ 0 \le t \le T,\ \ \vec{x}\,(0) = \vec{c}, \tag{48.1}$$

where A_n is a square matrix, and the problem is to find the G-estimator of the expression $\vec{b}^T\vec{x}\,(t) = \vec{b}^T\exp\{tA\}\,\vec{c},\ \ 0 \le t \le T$ when observation of matrix $X = A + \Xi$ is given and $\vec{b}$ is an n-dimensional vector. Let the entries of matrix Ξ satisfy the following conditions: $\xi_{ij},\ i > j,\ i, j = 1, \ldots, n$ are independent for every n,

$$\mathbf{E}\ \xi_{ij} = 0,\quad \mathbf{E}\ \xi_{ij}^2 = n^{-1},\quad \mathbf{E}\ \xi_{ij}\xi_{ji} = \rho n^{-1},\quad i \ne j.$$

Using estimators G_{34} we find the G_{48}-estimator of value $\vec{b}^T\vec{x}\,(t) = \vec{b}^T\exp\{tA_n\}\,\vec{c},\ \ 0 \le t \le T$;

$$G_{48} = \frac{1}{2\pi\mathrm{i}}\oint\limits_{|z|=\delta}\mathrm{e}^{tz}\left\{\frac{\partial}{\partial\gamma}\left[\frac{\vec{b}^T G_{34}(\alpha,\gamma)\vec{c}}{4\vec{b}^T G_{34}(\alpha,0)\vec{c}} - \mathrm{i}\frac{\vec{b}^T G_{34}(\alpha,\mathrm{i}\gamma)\vec{c}}{4\vec{b}^T G_{34}(\alpha,0)\vec{c}}\right]\right\}_{\gamma=0}\mathrm{d}z,$$

where

$$G_{34}(\alpha,\gamma) = \left\{I\alpha + (\Xi - zI + \gamma\vec{c}\vec{b}^T)^*(\Xi - Iz - \gamma\vec{c}\vec{b}^T)\right\}^{-1},$$

and $\delta > G_{32}^{\max}$, $\hat{\theta}\,(z)$ is the measurable solution of the equation

$$\hat{\theta}\,(z) + n^{-1}\mathrm{Tr}\,\left[\Xi - \hat{\theta}\,(z)\,I_n\right]^{-1} = z.$$

and

$$\vec{b}^T\vec{b} + \vec{c}^T\vec{c} \le c < \infty$$

then with probability one for every $t > 0$

$$\lim_{n\to\infty}\left\{G_{48} - \vec{b}^T\exp\left(tA_n\right)\vec{c}\right\} = 0.$$

49. G_{49}**–ESTIMATOR FOR SOLUTION OF THE SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH SYMMETRIC BLOCK STRUCTURE**

Suppose that

$$\frac{\mathrm{d}\vec{x}\,(t)}{\mathrm{d}t} = A_n\vec{x}\,(t)\,,\ \ 0 \le t \le T,\ \ \vec{x}\,(0) = \vec{c}, \tag{49.1}$$

where A_n is a symmetric matrix and the problem is to find the G-estimator of the expression $\vec{b}^T\vec{x}\,(t) = \vec{b}^T\exp\{tA\}\,\vec{c},\ \ 0 \le t \le T$, when observation of matrix $X = A + \Xi$ is given, $\vec{b}$ is an n-dimensional vector and the blocks of the matrix Ξ are independent. Using estimators G_8 we find the G_{45}-estimator of value $\vec{b}^T\vec{x}\,(t) = \vec{b}^T\exp\{tA_n\}\,\vec{c},\ \ 0 \le t \le T$:

$$G_{49} = \vec{b}\oint\limits_{|z|=\delta}\exp\left(tz\right)\left[X_n - \hat{\theta}\,(z)\right]^{-1}\vec{c}\,\mathrm{d}z,$$

where $\delta > G_{32}^{\max}$, $A_{pq\times pq} = \left(A_{ks}^{(n)}\right)_{k,s=1}^{p}$, $A_{ks}^{(n)} = A_{ks}^{(n)T}$ and $A_{ks}^{(n)}$; $k \geq s$, $k,s = 1,...,p$ are blocks of the dimension q, and let $\vec{x}$, $\vec{b}$ be vectors.

$$\vec{d}^T \vec{G}_8 = -\mathrm{Re}\ \left[X_{pq\times pq} + C\left(\varepsilon\right) + \mathrm{i}\varepsilon I_n\right]^{-1} \vec{b},$$

where $X_{pq\times pq}$ is an observation of matrix $\Xi_{pq\times pq} + A_{pq\times pq}$, $\Xi_{pq\times pq} = \left(\Xi_{ks}^{(n)}\right)_{k,s=1}^{p}$, $\Xi_{ks}^{(n)} = \Xi_{ks}^{(n)T}$ and $\Xi_{ks}^{(n)}$; $k \geq s$, $k.s = 1,...,p$ are independent random blocks of the dimension q, $C_{pq\times pq}\left(\varepsilon\right) = \left(\delta_{ij} C_{jj}^{(n)}\left(\varepsilon\right)\right)_{i,j=1}^{p}$ and the matrix-blocks $C_{ss}\left(\varepsilon\right)$ satisfy in the point $z = \mathrm{i}\varepsilon$ the canonical equation

$$C_{jj}\left(\varepsilon\right) = \mathrm{Re}\,\mathbf{E}\sum_{s=1}^{p}\Xi_{js}^{(n)} Q_{ss}\ \Xi_{js}^{(n)T}\Big|_{Q=\left[X_{pq\times pq}+C_{pq\times pq}(\varepsilon)+\mathrm{i}\varepsilon I_n\right]^{-1}}.$$

It is proven in [Gir84]] that under certain conditions, for every $\gamma > 0$

$$\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\mathbf{P}\left\{\left|\vec{d}^T\left(\vec{G}_8 - \vec{x}_\varepsilon\right)\right| > \gamma\right\} = 0.$$

50. G_{50}-CONSISTENT ESTIMATOR FOR SOLUTION OF LINEAR PROGRAMMING PROBLEM (LPP)

This section is devoted to the main G-estimator for the solutions of the LPP. Consider a standard deterministic LPP:

$$\max_{\vec{x}:A\vec{x}\leq\vec{b},\ \vec{x}\geq\vec{0},\ \vec{x}\in R^n}\vec{c}^T\vec{x}.$$

In some applied problems, the vectors $\vec{c}$ and $\vec{b}$ are known, but matrix a is unknown, but sample observations are available.

We will formulate the LPP as follows: find

$$\inf_{\vec{u}\in M}\vec{c}^T A^{-1}\left(\vec{b}+\vec{u}\right) = \vec{c}^T A^{-1}\left(\vec{b}+\vec{u}^*\right),$$

where

$$M = \left\{\vec{u}:\ \left\|A^{-1}\left(\vec{b}+\vec{u}\right)\right\| < 1;\ A^{-1}\left(\vec{b}+\vec{u}\right) > 0\right\},$$

when instead of a square unknown symmetric matrix $A = (a_{ij})$ of order n we have the observation of a random symmetric matrix $\Xi = (\xi_{ij})$, whose entries $\xi_{ij}^{(n)}$, $i \geq j$, $i,j = 1,...,n$ are independent for every n,

$$\mathbf{E}\,\xi_{ij}^{(n)} = a_{ij}^{(n)},\ \mathbf{Var}\,\xi_{ij}^{(n)} = \sigma_{ij}^{(n)}, i \geq j,\ i,j = 1,...,n,$$

$$\sup_n \max_{i=1,\ldots,n}\sum_{j=1}^{n}\sigma_{ij}^{(n)} < \infty.$$

In the general case, the system of equations $\Xi\vec{x} = \vec{b}+\vec{u}$ for the fixed vector $\vec{u}$ does not have any solutions, or has an uncountable number of solutions. We choose an estimator of the solution of this system in the regularized form:

$$\vec{x}_{\varepsilon}^{(n)} = \mathrm{Re}\ [\mathrm{i}\varepsilon I_n + Y_n + \Xi_n]^{-1} \left(\vec{b} + \vec{u}\right),$$

where $Y_n = \{\gamma_i \delta_{ij},\ \ i,j = 1,...,n\}$ is the diagonal matrix of order n, $\gamma_1, \gamma_2, ..., \gamma_n$ are the real solutions of the system of equations

$$\varphi_i \{Y_n\} = 0,\ \ i = 1,...,n;$$

where

$$\varphi_j \{Y_n\} = \gamma_j - \mathrm{Re} \sum\nolimits_{k=1}^{n} \sigma_{kj}^{(n)} \left\{(Y_n + \mathrm{i}\varepsilon I_n + \Xi_n)^{-1}\right\}_{kk},$$

$\varepsilon \neq 0$ is a real parameter. Let us introduce the G_{50}-estimator :

$$G_{50}(\varepsilon, n) = \inf_{\vec{u} \in L} \vec{c}^T \mathrm{Re}\ [\mathrm{i}\varepsilon I_n + Y_n + \Xi_n]^{-1} \left(\vec{b} + \vec{u}\right),$$

$$L = \Big\{\vec{u} :\ \vec{u} \leq 0;\ \left\| \mathrm{Re}\ [\mathrm{i}\varepsilon I_n + Y_n + \Xi_n]^{-1} \left(\vec{b} + \vec{u}\right)\right\| \leq 1\ ;$$
$$\mathrm{Re}\ [\mathrm{i}\varepsilon I_n + Y_n + \Xi_n]^{-1} \left(\vec{b} + \vec{u}\right) > 0\Big\}$$

of the expression $\vec{c}^T A^{-1} \left(\vec{b} + \vec{u}^*\right)$. Using the proof of Theorem 8.1 from Chapter 7 [Gir84] we get

THEOREM 50.1. *Let*

$$\sup_n \max_{i=1,\dots,n} \sum_{j=1}^{n} \left\{|a_{ij}| + \sigma_{ij}^{(n)} + |c_j| + |b_j|\right\} < \infty,$$

$$\lim_{N\to\infty} \lim_{n\to\infty} \sum_{j=N}^{n} \{|a_{ij}| + |c_j| + |b_j|\} < \infty,$$

and let Lindeberg's condition hold : for every $\tau > 0$

$$\lim_{n\to\infty} \max_{i=1,\dots,n} \sum_{j=1}^{n} \mathbf{E} \left[\xi_{ij}^{(n)} - a_{ij}^{(n)}\right]^2 \chi \left\{\left|\xi_{ij}^{(n)} - a_{ij}^{(n)}\right| > \tau\right\} = 0,$$

$$\sup_n \max_{i=1,\dots,n} \sum_{j=1}^{n} \sigma_{ij}^{(n)} \lambda_j^{-2} < 1,$$

where $\lambda_1 \geq \cdots \geq \lambda_n$ *are eigenvalues of the matrix* A, *and*

$$\inf_n |\lambda_n| > 0.$$

Then

$$\lim_{\varepsilon\to 0} p \lim_{n\to\infty} \left[G_{50}(\varepsilon, n) - \vec{c}^T A^{-1} \left(\vec{b} + \vec{u}^*\right)\right] = 0.$$

51. G_{51}-ESTIMATOR OF SOLUTIONS OF LPP

Using some results from [Gir84] (Chapter 11, Section 17; Chapter 13, Section 11) we find the G_{51}-consistent estimator for the solution of the LPP, i.e., we obtain the estimator under the conditions of general statistical analysis:

$$G_{51}(\varepsilon, n) = \min_{\vec{u}\in L} \vec{c}^T B\left(\vec{b}+\vec{u}\right),$$

where

$$B = \mathrm{Re}\left\{I\left(\hat{\theta}_1 + \mathrm{i}\varepsilon\right) + (A+\Xi)^T (A+\Xi)\right\}^{-1} (A+\Xi)^T$$

$$L = \left\{\vec{u}:\ \vec{u} \le 0, B(\vec{b}+\vec{u}) \ge 0,\ (\vec{b}+\vec{u})^T B^T B(\vec{b}+\vec{u}) \le 1\right\},$$

$\hat{\theta}_1 \ge \hat{\theta}_2 \ge \cdots \ge \hat{\theta}_k \ge \cdots$ are measurable real solutions of the equation $f_n(\theta) = \alpha$, where

$$f_n(\theta) = \theta \mathrm{Re}\,[1+\delta_1 a(\theta)]^2 - \varepsilon \mathrm{Im}\,[1+\delta_1 a(\theta)]^2 + (\delta_1 - \delta_2)[1+\delta_1 \mathrm{Re}\, a(\theta)],$$

$$a(\theta) = n^{-1}\mathrm{Tr}\left[I(\theta+\mathrm{i}\varepsilon) + (A+\Xi)^T (A+\Xi)\right]^{-1},$$

$$\delta_1 = \sigma_n^2 n,\ \ \delta_2 = \sigma_n^2 m.$$

THEOREM 51.1. *For any $n = 1, 2, \ldots$, let the entries $x_{pl}^{(n)},\ p = 1, \ldots, n,\ \ l = 1, \ldots, m$ of the matrix X be independent,*

$$\mathbf{E}\, x_{pl}^{(n)} = a_{pl}^{(n)},\ \ \mathbf{Var}\, x_{pl}^{(n)} = \sigma_n^2.$$

Let the generalized G-condition be fulfilled:

$$\limsup_{n\to\infty} \sigma_n^2 n s_n^{-1} = c_1 < \infty,\ \limsup_{n\to\infty} \sigma_n^2 m_n s_n^{-1} = c_2 < \infty,\ \limsup_{n\to\infty} m_n n^{-1} = c_3 < 1,$$

$\lambda_m + \alpha > h > 0$, *where* $\lambda_1 \ge \cdots \ge \lambda_m$ *are eigenvalues of the matrix* $A^T A$*;*

$$\limsup_{n\to\infty}\left[\vec{b}^T\vec{b} + \sup_{k=1,\ldots,m} \vec{a}_k^T \vec{a}_k\right] < \infty,$$

where $\vec{a}_k$ are columns of the matrix A,

$$\sup_n \lambda_1 < \infty,$$

for a certain $\delta > 0$

$$\sup_n \max_{\substack{p=1,\ldots,n,\\ l=i,\ldots,m}} \mathbf{E}\left|\sqrt{n}\left[x_{pl}^{(n)} - a_{pl}\right]\right|^{4+\delta} < \infty,$$

$$1 - 2\tau - 4^{-1}\tau^2 > 0,$$

where $\tau = \delta_1 n^{-1} \sum_{k=1}^{m} (\alpha + \lambda_k)^{-1}$ *and*

$$\limsup_{n\to\infty} \frac{\delta_1 - \delta_2 + 2\alpha}{1 - 2\tau - 4^{-1}\tau^2} \delta_1 n^{-1} \sum\nolimits_{k=1}^{m} (\alpha + \lambda_k)^{-2} < 1.$$

Then

$$\lim_{\varepsilon\to 0} p \lim_{n\to\infty} \left| G_{51}(\varepsilon, n) - \min_{\vec{u}\in R} \vec{c}^T \left\{ I\alpha + A^T A \right\}^{-1} A^T (\vec{b} + \vec{u}) \right| = 0,$$

where

$$R = \left\{ \vec{u} : \ \vec{u} \le 0, \ Q\left(\vec{b} + \vec{u}\right) \ge 0, \ \left(\vec{b} + \vec{u}\right)^T Q^T Q \left(\vec{b} + \vec{u}\right) \le 1 \right\},$$

$$Q = \left(I\alpha + A^T A\right)^{-1} A^T.$$

52. G_{52}-ESTIMATOR OF SOLUTION OF LPP WITH NONSYMMETRIC MATRIX A

From the previous Section (see also [Gir84]) it follows that the LPP can be formulated in the following form: find

$$\min_{\vec{u}\in L} \vec{c}^T \left(\alpha I + A^T A\right)^{-1} A^T \left(\vec{b} + \vec{u}\right),$$

where

$$L = \left\{ \vec{u} : \ \vec{u} \le 0; \ \hat{B}\left(\vec{b} + \vec{u}\right) \ge 0; \ \left(\vec{b} + \vec{u}\right)^T \hat{B}^T \hat{B} \left(\vec{b} + \vec{u}\right) \le 1 \right\},$$

$$\hat{B} = \left(\alpha I + A^T A\right)^{-1} A^T.$$

Therefore, by Theorem 6.1 from Chapter 6 [Gir84], the following statement is valid. Consider the G-estimator:

$$G_{52}(\varepsilon, n) = \min_{\vec{\theta}\in M} \vec{c}^T \left[C_1 + \mathrm{i}\varepsilon I_m + X^T (C_2 - \mathrm{i}\varepsilon I_n)^{-1} X \right]^{-1} \times X^T (C_2 - \mathrm{i}\varepsilon I_n)^{-1} \left(\vec{b} + \vec{\theta}\right),$$

where $C_1 = (c_{1i}\delta_{ij})_{i,j=1}^{m}, C_2 = (c_{2i}\delta_{ij})_{i,j=1}^{n}$ are diagonal matrices, whose diagonal entries satisfy the canonical equation:

$$\begin{cases} c_{1p} = \alpha + \sum\limits_{j=1}^{n} \sigma_{jp}^{(n)} \left\{ \left[(\delta_{ij}(c_{2i} - \mathrm{i}\varepsilon)) + X (\delta_{ij}(c_{1i} + \mathrm{i}\varepsilon))^{-1} X^T \right]^{-1} \right\}_{jj}; \ p = 1, ..., m, \\ c_{2k} = 1 + \sum\limits_{j=1}^{m} \sigma_{kj}^{(n)} \left\{ \left[(\delta_{ij}(c_{1i} + \mathrm{i}\varepsilon)) + X^T (\delta_{ij}(c_{2i} - \mathrm{i}\varepsilon))^{-1} X \right]^{-1} \right\}_{jj}, \ k = 1, ..., n. \end{cases},$$

$$M = \left\{ \vec{\theta} : , \vec{\theta} \le 0, \ R\left(\vec{b} + \vec{\theta}\right) \le 0, \ \left(\vec{b} + \vec{\theta}\right)^T R^T R \left(\vec{b} + \vec{\theta}\right) \le 1 \right\},$$

and

$$R = \left[C_1 + \mathrm{i}\varepsilon I_m + X^T \left(C_2 - \mathrm{i}\varepsilon I_n\right)^{-1} X\right]^{-1} X^T \left(C_2 - \mathrm{i}\varepsilon I_n\right)^{-1}.$$

THEOREM 52.1. *Let Ξ be a random matrix of the size $n \times m$; $m \le n$, whose entries $\xi_{ij}^{(n)}$; $i = 1, ..., n$; $j = 1, ..., m$ are independent for every n,*

$$\mathbf{E}\,\Xi = A = (a_{ij})_{i=1,\dots,n}^{j=1,\dots,m};\ \mathbf{E}\,\xi_{kp}^{(n)} = a_{kp}^{(n)};$$

$$\mathbf{E}\left[\xi_{kp}^{(n)} - a_{kp}^{(n)}\right]^2 = n^{-1};\quad k = 1, ..., n;\ p = 1, ..., m,$$

for a certain δ

$$\sup_n \max_{\substack{i=1,\dots,n;\\ j=1,\dots,m}} \mathbf{E}\left|\left(\xi_{ij}^{(n)} - \mathbf{E}\,\xi_{ij}^{(n)}\right)\sqrt{n}\right|^{4+\delta} < \infty,$$

$$\sup_n \max_{\substack{k=1,\dots,m;\\ l=1,\dots,n}} \left[\sum_{j=1}^{n}\left[|a_{jk}| + |b_j|\right] + \sum_{j=1}^{m}|c_j| + |a_{lj}|\right] < \infty;\quad \liminf_{n\to\infty} c^T c > 0$$

and (see Section 50)

$$\liminf_{n\to\infty} \beta_{1n} > 0.$$

Then

$$\lim_{\varepsilon\to 0} p\lim_{n\to\infty}\left|G_{52}(\varepsilon, n) - \min_{\vec{u}\in L}\vec{c}^T B\left(\vec{b} + \vec{u}\right)\right| = 0,$$

where $B = \left[I\alpha + A^T A\right]^{-1} A^T$, $\alpha > 0$,

$$L = \left\{\vec{u}, \vec{u} \le 0,\ B(\vec{b} + \vec{u}) \ge 0,\ (\vec{b} + \vec{u})^T B^T B(\vec{b} + \vec{u}) \le 1\right\}.$$

53. G_{53}-ESTIMATOR OF SOLUTION OF LPP OBTAINED BY INTEGRAL REPRESENTATION METHOD

Our review would be incomplete if we did not mention the integral representation method. In this section, under rather general assumptions and by means of integral representations for determinants, we formulate the limit theorem for the distribution of $\vec{x}_n^*$ of the equation

$$\min_{\vec{x}_n:\ A_n(\omega)\vec{x}_n \le \vec{b}_n(\omega);\ \vec{x}_n \ge 0} \mathbf{E}\, f\left\{\vec{x}_n^T \vec{\xi}_n(\omega)\right\} = \mathbf{E}\, f\left\{\vec{x}_n^{*T}\vec{\xi}_n(\omega)\right\}$$

where $A_n(\omega)$ is a random $n \times n$ matrix, $\vec{x}_n$, $\vec{\xi}_n(\omega)$, $\vec{b}_n(\omega)$ are random vectors and f is a measurable function. The basic result is that under certain conditions, the matrix $A_n(\omega)$ can be replaced by the approximate matrix which has only diagonal random entries equal to some sums of entries of the matrix $A_n(\omega)$. If the law of large numbers holds for these sums, then the diagonal entries can be replaced by deterministic values. The obtained result makes it possible to simplify the calculation of the solution $\vec{x}_n^*$ considerably, as well as to reduce the original stochastic problem to a deterministic one under certain conditions. Assume, that we have to solve the following linear stochastic problem: find

$$\inf \mathbf{E} f\left(\vec{x}_n,\ \vec{\xi}_n(\omega)\right) = \inf_{G\in L} \int f(\vec{u}_2,\ \vec{u}_1)\, \mathrm{d}G(\vec{u}_1,\ \vec{u}_2)$$

on the distribution function set

$$\begin{aligned} L &= \{G(\vec{u}_1,\ \vec{u}_2)\} \\ &= \left\{\mathbf{P}\left[\vec{x}_n(\omega) < \vec{u}_1,\ \vec{\xi}_n(\omega) < \vec{u}_2,\ \|\vec{x}_n(\omega)\| \le 1,\ \vec{x}_n(\omega) \ge 0\right],\ \ \vec{v} \le 0\right\}, \end{aligned}$$

where $\vec{x}(\omega)$ is a solution of the system of equation

$$\{I + A_n(\omega)\}\, \vec{x}_n(\omega) = \vec{\eta}_n(\omega) + \vec{v},$$

$A_n = \left(\xi_{ij}^{(n)}\right)$ is a random matrix of order n; $\vec{x}_n(\omega)$, $\vec{\eta}_n(\omega)$, $\vec{v}$ are nonrandom vectors, f is a certain measurable function chosen in such a way that there exists the integral $\mathbf{E} f\left(\vec{x}_n(\omega),\ \vec{\xi}_n(\omega)\right)$ and random vectors $\vec{\xi}_n(\omega)$, $\vec{\eta}_n(\omega)$ do not depend on matrix $A_n = \left(\xi_{ij}^{(n)}\right)$.

Consider the G-estimator:

$$G_{53} = \inf_{\vec{v}_n \le 0,\ G\in L} \int f(\vec{u}_2,\ \vec{u}_1)\, \mathrm{d}G(\vec{u}_1,\ \vec{u}_2),$$

where

$$L = \left\{\left[\vec{y}_n(\omega) < \vec{u}_1,\ \vec{\xi}_n(\omega) < \vec{u}_2,\ \|\vec{y}_n(\omega)\| \le 1,\ \vec{y}_n(\omega) \ge 0\right],\ \ \vec{v} \le 0\right\}$$

and $\vec{y}_n(\omega)$ is a solution of the system of equations

$$\left\{I_n + \operatorname{diag}\left[\sum_{p\in T_i\cup K_i} \nu_{pi}\nu_{ip},\ \ i = 1, ..., n\right] + X_n\right\} \vec{y}_n = \vec{\eta}_n(\omega) + \vec{v}_n.$$

Here T_i and K_i are certain sets [Gir54, p. 248].

Using Theorem 8.4.1 from [Gir54, p.248] and the proofs of the previous theorems we get

Theorem 53.1. *Let the vectors*

$$\left(\xi_{ij}^{(n)}, \xi_{ji}^{(n)}\right),\ \ i \ge j;\ \ i, j = 1, ..., n$$

be independent for every n, and asymptotically constant,

$$\lim_{h\to\infty} \lim_{n\to\infty} \mathbf{P}\left\{\left|\sum_{i=1}^{n} \nu_{ii}^{(n)}\right| + \sum_{i,j=1}^{n} \left[\nu_{ij}^{(n)}\right]^2 \ge h\right\} = 0,$$

$$\sup_n \left[|\mathrm{Tr}\, B_n| + \mathrm{Tr}\, B_n B_n^T\right] < \infty,$$

$$\nu_{ij}^{(n)} = \xi_{ij}^{(n)} - a_{ij}^{(n)} - \rho_{ij}^{(n)},\ \rho_{ij}^{(n)} = \int_{|x|<\tau} x \, \mathrm{d}\mathbf{P}\, \xi_{ij}^{(n)} - a_{ij}^{(n)} < x,$$

$$b_{ij} = \rho_{ij} + a_{ij},\ \ B_n = (b_{ij}),$$

$\tau > 0$ an arbitrary constant,

$$\lim_{h\downarrow 0} \liminf_{n\to\infty} \mathbf{P}\left\{|\det A_n| > h\right\} = 1, \quad \lim_{h\to\infty} \lim_{n\to\infty} \mathbf{P}\left\{\left\|\vec{\xi}_n\right\| + \|\vec{\eta}_n\| \geq h\right\} = 0,$$

$$\underset{n\to\infty}{p\lim} \left[\sum_{p\in T_i\cup K_i} \nu_{pi}\nu_{ip} - c_i\right] = 0,\ \ i = 1, ..., n,$$

where c_p are certain nonrandom constants, and the function $|f(\vec{u}_2,\ \vec{u}_1)|$ is bounded by a nonrandom constant. Then

$$G_{53} = \inf_{F\in R} \int f(\vec{u}_2,\ \vec{u}_1)\, \mathrm{d}F(\vec{u}_1,\ \vec{u}_2) + 0(1),$$

where F is a set of distribution functions

$$\begin{aligned} L &= \{F(\vec{u}_1,\ \vec{u}_2)\} \\ &= \left\{\mathbf{P}\left[\vec{y}_n(\omega) < \vec{u}_1,\ \vec{\xi}_n(\omega) < \vec{u}_2,\ \|\vec{y}_n(\omega)\| \leq 1,\ \vec{y}_n(\omega) \geq 0\right],\ \ \vec{v} \leq 0\right\} \end{aligned}$$

and $\vec{y}_n(\omega)$ is a solution of the system of equations

$$\{I_n + B_n\}\, \vec{y}_n(\omega) = \vec{\eta}_n(\omega) + \vec{v}_n.$$

54. G_{54}-ESTIMATOR OF SOLUTION OF LPP WITH BLOCK STRUCTURE

Suppose that instead of a matrix A we have a block matrix $A+X$ of the size $p_1q_1\times p_2q_2$, where $\Xi_{ks}^{(n)}, k = 1, ..., p_1;\ s = 1, ..., p_2$ are independent, $\mathbf{E}\,\Xi_{ij}^{(n)} = 0$, $\mathbf{E}\left\|\Xi_{ij}^{(n)}\right\|^2 < \infty$.

From Section 52 it follows that the LPP can be formulated in the following form: find

$$\min_{\vec{u}\in L} \vec{c}^T \left(\alpha I + A^T A\right)^{-1} A^T \left(\vec{b} + \vec{u}\right),$$

where

$$L = \left\{\vec{u}:\ \ \vec{u} \leq 0;\ B\left(\vec{b} + \vec{u}\right) \geq 0;\ \left(\vec{b} + \vec{u}\right)^T B^T B \left(\vec{b} + \vec{u}\right) \leq 1\right\},$$

$$B = \left(\alpha I + A^T A\right)^{-1} A^T.$$

By virtue of Theorem 5.1 [Gir84, Chapter 8] and Theorem 5.1 [Gir84, Chapter 10] the following statement is valid [Gir84]. We introduce the following G-estimator:

$$G_{54} = \min_{\vec{\theta}\in M} \vec{c}^T \left[C_1 + \mathrm{i}\varepsilon I_m + X^T (C_2 - \mathrm{i}\varepsilon I_n)^{-1} X\right]^{-1} X^T (C_2 - \mathrm{i}\varepsilon I_n)^{-1} \left(\vec{b} + \vec{\theta}\right),$$

where

$$M = \left\{\vec{\theta} : , \vec{\theta} \le 0,\ R\left(\vec{b} + \vec{\theta}\right) \le 0,\ \left(\vec{b} + \vec{\theta}\right)^T R^T R \left(\vec{b} + \vec{\theta}\right) \le 1\right\},$$

$$R = \left[C_1 + \mathrm{i}\varepsilon I_m + X^T (C_2 - \mathrm{i}\varepsilon I_n)^{-1} X\right]^{-1} X^T (C_2 - \mathrm{i}\varepsilon I_n)^{-1},$$

where A is a matrix of size $np \times mq$, $n \ge m$, $\alpha > 0$ is a parameter of regularization,$\varepsilon > 0$; $\vec{b} \in R^{np}$; $\vec{d} \in R^{mq}$; X is an observations of the matrix $A + \Xi$, $\Xi = \left(\Xi_{ij}^{(n)}\right)_{i=1,\dots,n}^{j=1,\dots,m}$ is a random matrix with independent blocks $\Xi_{ij}^{(n)}$, $\mathbf{E}\,\Xi_{ij}^{(n)} = 0$, $\mathbf{E}\left\|\Xi_{ij}^{(n)}\right\|^2 < \infty$, and $C_1 = (C_{1i}\delta_{ij})_{i,j=1}^m, C_2 = (C_{2i}\delta_{ij})_{i,j=1}^n$ are block diagonal real matrices that are arbitrary measurable solutions of the system of nonlinear equations

$$C_{1l} + \mathrm{Re} \sum_{j=1}^{n} \left[\mathbf{E}\,\Xi_{jl}^{(n)T} \{Q_{jj}\} \Xi_{jl}\right]_{Q=\left[C_2 - \mathrm{i}\varepsilon I_n + X(C_1 + \mathrm{i}\varepsilon I_m)^{-1} X^T\right]^{-1}} = I\alpha;$$

$$C_{2k} + \mathrm{Re} \sum_{j=1}^{m} \left[\mathbf{E}\,\Xi_{kj}^{(n)} \{\Theta_{jj}\} \Xi_{kj}^T\right]_{\Theta=\left[C_1 + \mathrm{i}\varepsilon I_m + X^T (C_2 - \mathrm{i}\varepsilon I_n)^{-1} X\right]^{-1}} = I,$$

$$k = 1, ..., n;\ p = 1, ..., m.$$

Theorem 54.1. *Let*

$$\lim_{n\to\infty} \left[\max_{i=1,\dots,p_1} \sum_{j=1}^{p_2} \mathbf{E}\, \|\Xi_{ij}\|^2 + \max_{i=1,\dots,p_2} \sum_{j=1}^{p_1} E\,\|\Xi_{ji}\|^2\right] < \infty,$$

and let Lindeberg's condition be fulfilled: for any $\tau > 0$

$$\lim_{n\to\infty} \left\{\max_{i=1,\dots,p_1} \sum_{j=1}^{p_2} \mathbf{E}\, \|\Xi_{ij}\|^2 \chi\left(\|\Xi_{ij}\| > \tau\right) + \max_{i=1,\dots,p_2} \sum_{j=1}^{p_1} \mathbf{E}\, \|\Xi_{ji}\|^2 \chi\left(\|\Xi_{ji}\| > \tau\right)\right\} = 0,$$

$$\sup_{p_1,p_2} \left[\sum_{i=1}^{p_1 q_1} |b_i| + \sum_{i=1}^{p_2 q_2} |c_i|\right] < \infty,$$

and

$$\lim_{n\to\infty} \left[\max_{i=1,\dots,p_1} \sum_{j=1}^{p_2} |A_{ij}| + \max_{i=1,\dots,p_2} \sum_{j=1}^{p_1} |A_{ji}|\right] < \infty.$$

Then

$$\lim_{\varepsilon \to 0} p \lim_{n \to \infty} \left| G_{54} - \min_{\vec{u} \in L} \vec{c}^T B \left(\vec{b} + \vec{u} \right) \right| = 0,$$

where

$$B = \left[I\alpha + A^T A \right]^{-1} A^T,$$

$$L = \left\{ \vec{u} : \ \ \|\vec{u}\| \leq 1, \ \vec{u} \leq 0, \ B(\vec{b} + \vec{u}) \geq 0, \ (\vec{b} + \vec{u})^T B^T B (\vec{b} + \vec{u}) \leq 1 \right\}.$$